An explicit cutoff for the Erdős–Gyárfás balanced-coloring conjecture

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Abstract. Every r-coloring of the edges of K_{r^2+1} contains $r+1$ vertices missing a color whenever $r \geq 255 \cdot 2^{136}$. We prove this by counting the edges that can prevent an independent choice across monochromatic cliques. Lean verifies the result from the original coloring. A separate proof at $r = 12$, together with the fixed cases through eleven, leaves only $13 \leq r < 255 \cdot 2^{136}$ as possible exceptions.

The affine-plane construction

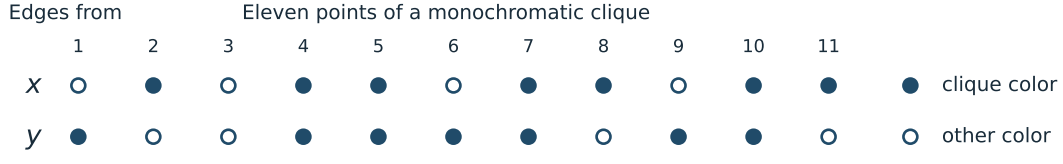
Call a coloring *balanced* if every set of $r+1$ vertices spans an edge of each color. For a prime power $r \geq 3$, take the r^2 points of an affine plane of order r. Color each edge by the direction of its line, then merge two of the $r+1$ direction classes. For each direction, the r parallel lines partition the plane, so two of any $r+1$ points determine that direction. The coloring is therefore balanced. Erdős and Gyárfás conjectured that no balanced r-coloring exists on r^2+1 vertices for any $r \geq 3$. They proved this for $r = 3, 4$. Original paper, Conjecture 1 and Lemmas 1–2

Theorem A. Let r be an integer with $r \geq 2^{144}$. For every coloring of the edges of K_{r^2+1} with r colors there is a set of $r+1$ vertices and a color that occurs on none of its edges.

Suppose the coloring is balanced and fix a color with the fewest edges. A stability theorem gives a partition in which most edges of that color lie within parts. We extract monochromatic cliques from the parts. To obtain $r+1$ vertices missing the chosen color, we seek one representative from each retained clique and additional vertices outside them. The edges between the cliques and the exterior constrain these choices. Counting the edges incident to deleted vertices also bounds how many parts can be discarded. In each of five parameter ranges, these bounds contradict balance. Appendix I gives the sharper cutoff $255 \cdot 2^{136}$. Proof of the sharper bound

Theorem B. In every twelve-coloring of the edges of K_{145}, some thirteen vertices miss a color.

At twelve colors the reduction gives a 61-vertex set spanning at most 408 edges of the chosen color. Under that bound, five disjoint eleven-vertex cliques remain. Each of the six exterior vertices has at least seven neighbors in some clique in the chosen color. Two exterior vertices must use the same clique. Among those thirteen vertices at most $4+4+1 = 9$ edges can carry the eleven other colors, contradicting balance. This proves the lower bound of 409. Bounds on 47- and 59-vertex sets exclude the remaining cases.



Each hollow circle marks a possible exception to adjacency in the chosen color.

Fixed cases and verification

Colors	Previous results
$3, 4$	Erdős and Gyárfás proved both cases in their 1999 paper. Original paper
$5, 6, 7, 8, 9$	Robert Sneiderman released the proofs for five through eight on 20 July 2026 and for nine on 21 July. The proofs at five and six are graph-theoretic. The others use finite certificates. Five through eight · Nine
$10, 11$	Adam Lee Terpstra directed the computer-assisted proofs produced with OpenAI Codex and released on 11 August 2026. We later obtained separate arguments. August release · Ten, Theorem 21 · Eleven
12	Theorem B in this manuscript. The local edge bounds and the final reduction have two independent integer calculations. Lower-bound calculation · Capacity calculation

Nick Winter's five-color Lean release is dated 13 July. Sneiderman posted his five- and six-color manuscripts on 18 July. Ramazan Kara based his five-color verification on Sneiderman's proof. Winter's six-color Lean proof credits Sneiderman's argument for its route and intermediate results. Winter, five colors · July manuscripts · Kara · Winter, six colors

Conner Silverstein and Anthony Rose also gave computational proofs at five colors in July. Hai Xia and Stefan Szeider submitted a separate five-color proof to arXiv on 17 October. Silverstein · Rose · Xia and Szeider

Formal verification. Given any coloring at $r \geq 255 \cdot 2^{136}$, Lean proves that a set of $r + 1$ vertices misses a color. Three kernels accepted the same proof export. At twelve colors, Lean verifies the final thirteen-vertex witness. The written argument and two integer calculations establish the hypotheses for that witness. Independent mathematical review is pending. Cutoff proof and replay · Twelve-color witness

Contributions. Robert Sneiderman directed the research and handles correspondence. OpenAI GPT-6.2 Astra, the AI model used in Codex, contributed to proof search, calculations, Lean proofs and drafting. Fable contributed fixed-row arguments and checked earlier steps. The July releases document the tools used for those earlier results. July disclosure

The twelve-color proof uses the two- and three-row bounds and the four-row certificates.

References. P. Erdős and A. Gyárfás, "Split and balanced colorings of complete graphs," *Discrete Mathematics* **200** (1999), 79–86. Z. Füredi gives the stability partition used for the sparsest color. A. Graf, D. G. Harris and P. Haxell give the weighted independent transversal theorem used to choose representatives from the rows. Erdős and Gyárfás · Füredi, Theorem 1 · Graf, Harris and Haxell, Theorem 7

The cutoff at 2^{144}

Theorem. For every integer $r \geq 2^{144}$, every r-coloring of the edges of K_{r^2+1} contains $r+1$ vertices missing a color.

We use the five bands, sampling constants and signed-cut inequality of the weighted transversal argument. The incidence count for first-stage deletions also bounds the number of unqualified destinations. This joint bound increases the contribution of the retained rows at the upper endpoint of the width integral. Exact checker

Lean proves that every feasible pair of real pruning parameters belongs to a band with width margin greater than $7/10000$. It checks the endpoint inequality for the reserved rows. The Lean proof guide describes the row extraction and width integral. Formal width comparisons

1. An upper bound on omitted destinations

Retain the preceding notation. The raw parts R_i have size $r+b_i$. Let U_i be the number of their internal nontarget pairs minus $(r-1)b_i$. Then $U_i \geq 0$, $\sum U_i \leq r^2/2$, and $\sum (b_i)_+ \leq r/2$. A destination qualifies when $b_i \leq r/256$ and $U_i \leq r^2/1200$. In the first stage, exactly pr vertices are deleted. Assign each to its recorded qualifying destination. At one such destination the preceding star count bounds the number of assigned sources by $U_i/((61/250)r)$. Summing only over qualifying destinations shows that they consume at least $(61/250)pr^2$ of the total U_i.

Consequently at most $600-(1464/5)p$ destinations fail the U_i test. At most 128 fail the size test. With n the combined number of short and chosen lists, the initial number e_0 of omitted slots satisfies

$$e_0 + 1 \leq 729 - (1464/5)p + n.$$

Here $0 \leq p \leq 125/61$, $n \leq 1411$, $K = 1120$, $K' = K + 2n$. In the previous width argument set aside the $K' - e_0 - 1$ heaviest rows. Their number, divided by K', is at least

$$\lambda_n = \frac{K\lambda + n}{K + 2n}, \qquad \lambda = \frac{391 + (1464/5)p}{K}.$$

In particular $0 < \lambda < 1$, $\lambda_n \geq \lambda - n/K$, and $\lambda_n \geq \min(\lambda, 1/2)$.

2. The upper endpoint of the width integral

In one of the five preceding bands let b, α, β be its constants. Set $S = 2^{(144-b)/4}$, $\ell = \alpha/S$, and $H = \alpha/K'$. The checker verifies $S > 8K'$ throughout. Retain the row weights a_i, the nonnegative coefficients $Q_i \leq a_i^2/\alpha$ and the function $F(z)$ of Appendix A, §3. Assume $F(z) \geq 1$ on $[\ell, H]$. Put $u = \max a_i/H$ over the rows still in the integral, with value zero for an empty family. The rows set aside contribute at least $\lambda_n u$ to the available budget, since each has weight at least uH and costs at least uH/α.

Write $A = \sum_i Q_i/H$ for the remaining rows. For $t \geq u$ and $t \leq 1$, the formula for the integral gives $F(tH) = t + A/t$. If $u \leq 1$, the integral of the remaining row terms above H, with respect to dz/z, is A. The integral below ℓ is at least one by the previous lower-endpoint argument. That unit pays exactly for the base term's integral on $[0, H]$. For any $h \in [\max(u, \ell/H), 1]$ we can now count both the tail A and the excess over one on $[hH, H]$. The additional saving, beyond the rows set aside, is at least

$$A + \int_h^1 (t + A/t - 1)\frac{dt}{t} = 1 - h + A/h + \log h. \tag{1}$$

When $u > 1$ the reserved rows alone still give $\lambda_n u$.

Write Y, D for the sums of the normalized row incidence and deficit bounds before choosing the width. The original estimate (3) gives

$$Y/\alpha + D/\beta \le C - \gamma n, \quad C = Y_*/\alpha + (p + t_0)/\beta, \quad \gamma = \frac{1 - 2\tau - 1/256}{2\beta},$$

where $t_0 r$ is the second-stage deletion count, $\tau = 199/400$ and $Y_* = 2 - p/2 - \tau t_0$. Switching from $\log(S/K')$ to $\log(S/K)$ uses at most $2n/K$. Thus every saving above is supplemented by at least $\epsilon n = (\gamma - 2/K)n$, where $K\epsilon \ge 3/5$ in every band.

We claim that failure throughout $[\ell, H]$ implies

$$\log(S/K) \le C - L(p), \qquad L(p) = \min\left(\lambda, \frac{4333 + 737p}{10000}\right). \tag{2}$$

- If $u \le 1/8$, failure at $H/2$ requires $A \ge 1/4$. With $h = 1/8$, (1) exceeds $3/4$, which is larger than $L(p)$.
- If $1/8 \le u \le 1/2$, use $h = u$ and $A \ge 1/4$. Since $K\epsilon \ge 1/2$, the reserved rows and ϵn contribute at least λu. The total saving is at least $1 - u + 1/(4u) + \log u + \lambda u$. Its minimum on $(0, 1/2]$ occurs at $u = 1/(2(1 + \sqrt{\lambda}))$, by differentiation, and equals

  $$G(\lambda) = 1 + \sqrt{\lambda} - \log(2(1 + \sqrt{\lambda})).$$

  Its derivative in λ is $1/(2(1 + \sqrt{\lambda}))$, so G is concave. Exact bounds at $p = 0, 125/61$ give $G(\lambda) > (4333 + 737p)/10000$ throughout.
- If $1/2 \le u \le 1$, failure at uH gives $A \ge u(1 - u)$. Using (1) and the reserved rows gives $2(1 - u) + \log u + \lambda_n u + \epsilon n$, a concave function of u. At $1/2$ it agrees with the preceding lower bound before amortizing n. At 1 it equals $\lambda_n + \epsilon n$. The latter bound also applies when $u \ge 1$.
- To check this last endpoint, for $\lambda \le 1/2$ use $\lambda_n \ge \lambda$. Otherwise $\lambda_n \ge \max(1/2, \lambda - n/K)$, and minimizing these two affine bounds after adding $3n/(5K)$ gives $\lambda_n + \epsilon n \ge 1/2 + (3/5)(\lambda - 1/2)$. This exceeds the affine term in (2) at both ends of the interval $1/2 \le \lambda \le \lambda(125/61)$, and hence throughout.

3. The five comparisons

The pruning domain is $0 \le p \le 125/61$, $t_0 \ge 0$, $p/4 + \tau t_0 \le 1$. Clip it by each of the five band intervals for Y_*. For each of the two affine terms in (2), $\log(S/K) - C + L$ with that term substituted for L is affine on the clipped polygon. Testing all its vertices gives the following strict lower bounds for the minimum of the two comparisons.

Band for Y_*	b	Certified margin exceeds
0 to 0.50	14	0.002030
0.50 to 0.55	14	0.001133
0.55 to 0.68	14	0.010032
0.68 to 1	15	0.130541
1 to 2	17	0.460849

Thus (2) fails in every band. There is a width satisfying the normalized cut and deficit bounds (4) of Appendix A with $K \le s < S$ and $2^b s^4 < 2^{144} \le r$. The shortage argument and the signed cut then apply. All 1,280 exact signed-cut endpoints exceed 0.0003327 with their original parameters. □

At exponent 143 the present sufficient comparison at $p = 125/61$, $p/4 + \tau t_0 = 1$ has margin below -0.17125 in the first band. Improving the constants, the saving in (2), or the feasible domain could change this

comparison. Whether a coloring realizes that boundary is open. Rational endpoint check

The twelve-color theorem

Theorem. Every twelve-coloring of the edges of K_{145} has thirteen vertices whose induced complete graph misses a color.

In the five-row case, the 61 vertices span at least 409 edges of the target color. The global coloring permits at most 408. Two further local bounds, 283 edges on 47 vertices and 358 on 59 vertices, close the other cases. The proof uses earlier four-row certificates. The calculations have two integer implementations. Lean verifies the thirteen-vertex witness in Section 4. Forward calculation · Budget-first replay · Lean proof of the witness

1. Local edge bounds and neighborhood deletion

Call a coloring balanced if every thirteen vertices see all twelve colors. Suppose the coloring is balanced. Fix a target color a and use its graph for degrees, neighborhoods, cliques and independence. A set of type (s, n) has n vertices, independence number at most s, and no target K_{12}. Write $f(s, n)$ for a lower bound on its target edge count. The earlier order bounds on the size of such sets are $N(0) = 0$, $N(1) = 11$, $N(2) \le 23$, $N(3) \le 35$, $N(4) \le 48$. Only upper bounds are needed. The last two follow from the three-row theorem and $P(4)$ at twelve. Two- and three-row results, Sections 6 and 8

The 48-vertex input. In a set of type $(4, 48)$ every degree is at least twelve by $N(3)$. The four-row certificates at order twelve exclude every case in which the minimum total of $d - 12$ on an independent quartet is at most three. Section 2 of the same argument shows that either such a quartet exists or some other color partitions the 48 vertices into twelve independent quartets for the target. The latter partition gives a total degree of at least $48 \cdot 12 + 12 \cdot 4 = 624$. Thus $f(4, 48) \ge 312$. The 102 profiles surviving the earlier census all have quartet weight at least four. Four-row argument and census · Prerequisite verifier

In a set of type (s, n), deleting the closed neighborhood of a minimum-degree vertex leaves a set of type $(s-1, u)$, where $u = n-1-d$. Put $\tau = \max(0, u - N(s-2))$, using zero if the order bound is unavailable. A lower bound for the deletion cost is $\binom{d+1}{2} + P$, with

$$P = \begin{cases} 0 & d \le 10, \\ \max(1, \tau - 1) & d = 11, \\ \max(11, 2\tau - 6) & d = 12, \\ 11(d - 11) & d \ge 13. \end{cases} \tag{1}$$

Here is the argument behind the premiums. If A is the root's neighborhood, let m count the nontarget pairs in A, and let $j(v)$ count those incident to v. The number of target neighbors of $v \in A$ in the remainder is $j(v) + h_v$, with $h_v \ge 0$ by minimum degree. The exact deletion cost is $\binom{d+1}{2} + m + \sum h_v$. For any nontarget pair v, w in A, their target neighbors in the remainder meet every independent $(s-1)$-set there. Otherwise that set extends by v, w. A hitting set has size at least τ, so $j(v) + j(w) + h_v + h_w \ge \tau$. At $d = 11$ such a pair exists, since a complete A extends by the root, and $j(v) + j(w) \le m + 1$. At $d = 12$ the closed neighborhood sees all eleven other colors, all inside A, so $m \ge 11$. The missing pairs cannot form a star, which would leave a target K_{12} with the root, or lie on a triangle. Hence two are disjoint. Their four internal missing degrees sum to at most $m + 6$, giving the third line of (1). At higher degrees each other color needs at least $d - 11$ edges on the closed neighborhood, since deleting one endpoint per edge leaves an independent set in that color of size at most twelve. Deletion identity, Section 12

For each d, take the larger of $\lceil nd/2 \rceil$ and the residual floor plus (1)'s deletion cost, then minimize over d. The order bounds alone give $f(3, 35) \ge 208$ by this recursion. With the 48-vertex input, the only possible minimum degree at $(5, 61)$ with at most 408 edges is thirteen. Indeed $d \le 11$ leaves at least 49 nonneighbors and $d = 12$ costs at least $78 + 20$ above the 312 residual edges. The degree sum excludes $d \ge 14$.

2. Five private cliques and their colored cuts

Assume a set W of type $(5, 61)$ has at most 408 edges. Write $q(v) = d(v) - 13$. All q are nonnegative and $E = \sum q \le 2 \cdot 408 - 61 \cdot 13 = 23$. Any nontarget pair y, z has at most

$$2 + q(y) + q(z) \tag{2}$$

common target neighbors. Its common nonneighbors have independence at most three and hence order at most 35. Inclusion-exclusion gives (2).

At least 50 vertices have $q \le 1$, since each with $q \ge 2$ spends two units. By $N(4) \le 48$, choose five independent roots among these, minimizing their total weight k among such choices. Write $a_i = q(x_i) \in \{0, 1\}$. The root x_i and all points adjacent only to it among the roots form a private clique Q_i. Two nonadjacent private neighbors would replace x_i and give six independent vertices. Every member of Q_i has weight at least a_i by root replacement. Put $\ell_i = |Q_i| = 11 - b_i$, $B = \sum b_i$. Each of the $6 + B$ exterior vertices hits $j_y \ge 2$ roots. Root i meets $3 + b_i + a_i$ exterior vertices. Therefore

$$\sum_y (j_y - 2) = 3 + k - B, \qquad B \le 3 + k, \qquad \#\{y : j_y = 2\} \ge 3 + 2B - k. \tag{3}$$

Call vertices meeting exactly two roots ordinary. The private rows already spend at least $11k - B \ge 10k - 3$ units of excess. Thus $k \le 2,\ B \le 5$.

Let an ordinary vertex y meet roots i, j and miss p, t points of their respective rows. The two missed lists are completely target-joined. A nontarget pair across the lists, y, and the other three roots would be independent. Their union thus gives $p + t \le 11$. If C_{ij} is the actual number of target edges across the rows, then

$$pt \le C_{ij} \le 11 + b_i b_j, \qquad C_{ij} \le b_i b_j + p + t. \tag{4}$$

For the first upper bound each other color needs at least $\ell_i + \ell_j - 12$ edges on the union of the rows. All lie across the cut. After adding y, each needs one more edge, and only $p + t$ new nontarget edges are available. Subtracting these eleven requirements gives the second upper bound. The first upper bound holds for any pair of rows.

Assign y to the incident row where it misses fewer points, breaking ties arbitrarily. Its assigned miss count p is at most three. Indeed $(p-1)(t-1) \le 1 + b_i b_j \le 7$ by (4) and $B \le 5$. Two lists of length at least four violate this. Joining an eleven-point row completely would create a target K_{12}.

3. The 91 row profiles

The following finite inequality uses only (2)–(4) and the eleven other colors. For a candidate assigned row of size $\ell = 11 - b$, let D be its total q. Its root has weight a, so $\ell a \le D \le 23$. For each $p = 1, 2, 3$ try every other row with deficit c and every integer $p \le t < 11 - c$ satisfying $p + t \le 11,\ pt \le bc + p + t$. An empty list of choices forbids p. Otherwise the assigned vertex costs at least

$$\max\left(0,\ \min_{c,t}[\ell + 11 - c - p - t - 13],\ \ell - p - 2 - a - \left\lfloor \frac{D - \ell a}{p} \right\rfloor\right). \tag{5}$$

The first expression inside the minimum counts the target neighbors in the two rows. For the last term, every missed point has the $\ell - p$ hits as common neighbors with the assigned vertex. By (2) all p misses have weight at least $\ell - p - 2 - q(y)$. The row can spend at most $D - \ell a$ above its baseline. Full joins ($p = 0$) are permitted only for $b > 0$.

Enumerate the nondecreasing lists of misses on the row. Their length is at most $3 + b + a$ by the root

degree. For the z smallest misses we need

$$p_1 + \cdots + p_z + \binom{z}{2} \;\geq\; 11(z - b - 1). \tag{6}$$

Each other color needs at least $z - b - 1$ edges on the row and those z vertices. Only their misses into the row and their mutual pairs are available. Among g full joins, at most b can have weight less than $\lceil(\ell - 2)/2\rceil$. Any two below that threshold are target-adjacent by (2) because they share the entire row. Thus full joins cost at least $(g - b)_+\lceil(\ell - 2)/2\rceil$.

Charge the row its D, the cost (5) of each positive miss count, and this full-join bound. Each ordinary vertex is charged on just one row. Minimize over their numbers on the five rows, subject to at least $3 + 2B - k$ assignments. The table gives the minimum charge in this relaxation by k, B, over profiles with $\sum \ell_i a_i \leq 23$. The five pairs (b_i, a_i) are sorted only to remove permutations.

Root weight k	$B = 0$	$B = 1$	$B = 2$	$B = 3$	$B = 4$	$B = 5$	Profiles
0	21	25	29	28			7
1	24	28	31	28	24		26
2	28	31	34	31	27	29	58

Ninety profiles exceed 23. Only $k = B = 0$ remains. The second implementation instead splits a budget of 23 among the rows and maximizes how many ordinary vertices the rows can accept. It tests 434 choices with the roots of each weight labelled. Only the same profile survives, with capacity three. Minimum-charge verifier · Maximum-count verifier

4. Six exterior vertices and five cliques

All five rows now have eleven vertices. Two exterior vertices cannot both miss at most four points of the same row. That row and the two vertices would form a thirteen-set with at most $4 + 4 + 1 = 9$ edges available for the eleven nontarget colors.

In this profile an ordinary vertex misses one or two points of its assigned row, by (4) with $b_i = b_j = 0$. Consequently distinct ordinary vertices are assigned to distinct rows. Charge each its own weight and the weights of its missed points on the assigned row. These charges are disjoint. With p misses the charge is at least $q(y) + p\max(0, 11 - p - 2 - q(y))$, by (2). This is at least eight for $p = 1$ and at least seven for $p = 2$. Four ordinary vertices would spend at least 28. By (3) there are exactly three. The other three exterior vertices must each meet exactly three roots, since $\sum(j_y - 2) = 3$.

The three charges spend at least 21 of the 23 units. Each of the other exterior vertices therefore has weight at most two. In any row at most two of its points were charged, and the uncharged points in the entire graph have at most two units left. Every row thus contains at most four positive-weight points.

Take a vertex y meeting three roots. If it missed at least five points in each incident row, each missed list would contain a weight-zero point. Applying (2) to that point and y gives at most four hits in the row, so each missed list actually has at least seven members. Choose one uniformly and independently from each list. Each pair of rows has at most eleven target edges by (4), so the probability of any target pair is at most $33/49 < 1$. Some choice avoids the target. The three representatives, y, and the two unhit roots then give six independent vertices.

Hence each of the six exterior vertices misses at most four points of some row. No two can use the same row, and there are only five. This contradiction proves $f(5, 61) \geq 409$.

5. Edge bounds on 47 and 59 vertices

Two smaller bounds are needed. For $(4, 47)$ under a cap of 282, minimum degree below eleven contradicts $N(3)$. Degree eleven leaves 35 vertices and costs at least $66+11$ by (1), giving at least $208+77 = 285$. The degree sum now forces twelve-regularity. For $(5, 59)$ under a cap of 357, degree at most nine leaves more than 48 vertices. Degree ten costs at least $55 + 312$, and degree eleven costs at least $66 + 11 + 283 = 360$ once the 47-vertex bound is proved. A higher minimum degree than twelve exceeds the cap.

Handle these two cases in that order, now with $q = d - 12$. There are zero units of excess at 47 and at most six at 59. In each case choose s independent weight-zero roots. At 47 use $47 > N(3)$. At 59 at least 53 vertices have weight zero, more than $N(4)$. Inequality (2) again holds, by $N(2) \le 23$ at 47 and $N(3) \le 35$ at 59. There are $s-1+B$ exterior vertices and $\sum(j_y - 2) = 2 - B$, since now $n = 12s - 1$ and the root degree is twelve. Thus $B \le 2$ and at least $s - 3 + 2B$ vertices are ordinary. The same missed-list argument and (4) apply.

At 47, an ordinary vertex that misses any point of a row hits at most two of that row, by (2). Missing points in both incident rows would give at least seven misses per row, contradicting (4). Assign each ordinary vertex to a row it joins completely. All vertices assigned to the same row form a clique with it by (2), so at most B can be assigned. At least $1 + 2B$ were required. This proves $f(4, 47) \ge 283$.

At 59, temporarily discard points of weight above T, where $T = 2$ for $B \ge 1$ and $T = 3$ for $B = 0$. Each remaining row has more than $2 + 2T$ points. For $B \ge 1$, an ordinary vertex hitting at most six points of each incident row would have $pt \ge (5-b_i)(5-b_j) \ge 15 + b_i b_j$, contradicting (4). For $B = 0$, at most eight hits on both sides would give $p, t \ge 3$, violating $pt \le p + t$. Thus an ordinary vertex of weight at most T hits more than $2 + 2T$ original points of some row. By (2) it joins every retained point there. Its assigned companions are mutually adjacent by (2). Capacity is at most B plus the number discarded from the rows. Adding back the high-weight ordinary vertices gives at most $B + \lfloor 6/(T+1) \rfloor$ ordinary vertices in all, strictly below $2 + 2B$. This proves $f(5, 59) \ge 358$.

6. The global coloring

In a balanced K_{145}, the sparsest color has at most $\binom{145}{2}/12 = 870$ edges. By Proposition 7 it has a vertex of minimum degree $11 - t$ with $0 \le t \le 11$. Indeed, otherwise it is twelve-regular and Brooks' theorem gives a proper twelve-coloring of its target graph, hence an independent thirteen-set. No component can be a target K_{13}. Delete the closed neighborhood and repeatedly remove target K_{12}'s until none remain. Removing a full clique lowers the independence bound by one because every exterior vertex has at most one target neighbor on the clique. Every target-independent set of at most eleven vertices outside can therefore extend into it.

This leaves a set of order $n + t$, $n = 12s + 1$, $1 \le s \le 11$, with independence at most s and no target K_{12}. The first deletion costs at least $\binom{12-t}{2}$, and each of the $11 - s$ subsequent deletions costs at least 66. Sampling n vertices therefore gives a set of type (s, n) under the cap $870 - 66(12 - s)$. Explicitly, the checker compares

$$\left[870 - \binom{12-t}{2} - 66(11-s)\right]\binom{n}{2} \;\le\; [870 - 66(12-s)]\binom{n+t}{2}$$

at all 132 choices. Outer reduction, Section 6

The first four stops exceed the known order bounds. The remaining comparisons from (1), with floors at $(4, 48), (4, 47), (5, 59), (5, 61)$, are as follows.

Vertices	61	73	85	97	109	121	133
Target edge floor	409	475	541	608	675	742	809
Global cap	408	474	540	606	672	738	804

Vertices	61	73	85	97	109	121	133

With just the 48- and 61-vertex inputs, the same recurrence gives 474, 539 and 606 at the three later stops in question. The 47- and 59-vertex improvements supply the strict inequalities. All eleven stops are excluded, proving the theorem.

Reproduction. Run the following from the directory containing the scripts with Python 3.10 or newer. The four-row verifier reads the existing certificate files alongside it. The two order-twelve scripts use the standard library and import no code from one another.

```
python3 -B verify_four_row_descent.py 12
python3 -B verify_order_twelve.py
python3 -B replay_order_twelve.py
```

Appendix A. Weighted transversals and the width integral

This is the width argument at exponent 146. The cutoff chapter replaces its endpoint estimate.

1. The residual in the stability inequality

Suppose a counterexample exists. Choose a sparsest color J, and call its edges target edges. Each color has independence number at most r, and every monochromatic clique has at most r vertices. Write $p_k(n)$ for the minimum number of edges on n vertices with independence number at most k, the sum of the internal binomial coefficients of k balanced parts.

Consider any induced target graph with independence number at most k and excess $\eta = e_J - p_k(n)$. In its complement choose a vertex of maximum residual degree. Take that vertex and all its residual nonneighbors as a part, then continue on its neighbors. The pivots form a clique in the complement, so at most k parts occur. Pad with empty parts. In each step the residual degrees of the vertices in the part are at most the number of later vertices. Summing gives $e(\bar{J}) + \mu \le e(K(R_1, \dots, R_k))$, where μ counts internal pairs missing J. Let $V = \sum_i (|R_i| - r)^2$, let V_{bal} be its balanced value at the same n, k, and let X count the interpart target edges. Expanding binomial coefficients gives

$$\mu \le \eta - \tfrac{1}{2}(V - V_{\mathrm{bal}}), \qquad X = \eta + \mu - \tfrac{1}{2}(V - V_{\mathrm{bal}}) \le 2\eta - (V - V_{\mathrm{bal}}). \tag{1}$$

We also keep the inequality before summing. In part R_i the residual complement degrees sum to $2\mu_i + e_{\bar{J}}(R_i, R_{>i}) \le |R_i||R_{>i}|$, where $R_{>i}$ consists of all later parts. Thus there are at least $2\mu_i$ target edges directed from R_i to later parts. Füredi, proof of Theorem 1, equation (4)

On the full graph, take $k = r$. Then $V_{\mathrm{bal}} = 1$, $\eta \le \binom{r}{2}$ by sparsity, and $X/r^2 \le 1 - v$ for $v = V/r^2$. Write $b_i = |R_i| - r$, μ_i for the internal missing count, and $U_i = \mu_i - (r-1)b_i$. Each other color needs at least $(b_i)_+$ edges in R_i, by deleting one endpoint per edge. Hence $\sum_i (b_i)_+ \le r/2$, $U_i \ge 0$, and (1) gives $\sum_i U_i = \mu - (r-1) \le r^2(1-v)/2$.

A destination qualifies if $b_i \le r/256$ and $U_i \le r^2/1200$. At most 728 parts fail. First prune any vertex with more than $r/4$ current neighbors in one qualifying foreign part. Record one such destination and its current star. In a qualifying part the number of possible sources is at most $U_i/((61/250)r)$, hence at most ωr where $\omega = 5/1464$. This is the ordered-star count in the earlier extraction. Next prune while any current total interpart degree exceeds τr, with $\tau = 199/400$. Write pr, tr for the two deletion counts. The stars and the edges spent at the second stage are fresh. Consequently

$$p \le p_*(1-v), \qquad Z = 1 - p/4 - \tau t \ge v, \qquad p_* = 125/61.$$

Set $Y_* = 2 - p/2 - \tau t$. For any family of surviving cliques, charge each clique its target edges outside its raw part, excluding the recorded stars. If $y_i' r^2$ is this count at clique i, then $\sum y_i' \le Y_* - 2v$. Indeed the original incidence count is at most $2(1-v)r^2$. Both ends of a recorded edge disappear from the sum, and at least one end of each second-stage edge disappears. Edges between surviving cliques are fully counted. First-stage count · Recorded edges

2. Missing representatives in short lists

Before trimming any part, choose a maximal family of q pairs, one pair in each of q distinct surviving parts, with all $2q$ endpoints independent. Delete their neighbors in the unchosen parts. Each remaining list is a clique by maximality. Write $m_i \le r$ for its size. Set $d = 1 - 2\tau = 1/200$, $L = (1+2\tau)/2 = 399/400$, and $b = \lfloor Lr \rfloor$. Since $(L - 2\tau)r > 1$, we have $b > 2\lfloor \tau r \rfloor$.

For the following test alone, truncate lists of size at least b to b and pad shorter ones with isolated dummy vertices to b. Erase within-block edges. The resulting graph has maximum degree at most $\lfloor \tau r \rfloor$. Give

real vertices weight one and dummies weight zero. The weighted independent transversal theorem gives a choice of one vertex per block, with mutually independent real choices, of weight at least

$$r - q - W, \qquad W = \sum_{m_i < b} (1 - m_i/b).$$

The real choices and the $2q$ endpoints are independent in the original graph. The independence cap r implies $q \le W$. All lists, including those from unqualified parts, enter this test. Graf, Harris and Haxell, Theorem 7

Keep the original untruncated lists of size at least b in qualifying parts as ordinary cliques C_i. Put $x_i = 1 - |C_i|/r,\ D = \sum x_i,\ Y = \sum y'_i$. Let $a_i r$ be the early deletions in raw part i, during both pruning stages. The chosen parts then contain exactly $(q + \sum_{\text{chosen}} (b_i/r - a_i))r$ survivors. Their endpoints remove at most $2\tau q r$ further vertices from unchosen parts. The F short lists have combined deficit at least $(1 - L)F + LW$, since $m_i/r \le L m_i/b$. Using $q \le W \le F$, and dropping the favorable $-1/r$ from the original order, gives

$$D + \tfrac{d}{2}(F + q) \le p + t + \sum_{\text{chosen}} (b_i/r - a_i). \tag{2}$$

On $0 \le p \le p_*,\ t \ge 0,\ Z \ge 0$, the maximum of $1/2 + p + t$ is $85689/24278 < 3.53$. Hence $D < 3.53$ and $n = F + q \le 1411$. If e_0 is the number of omitted slots, then $e_0 \le 728 + n$.

Consider the chosen raw parts with $b_i > r/256$. Write $O = \sum b_i/r$ and $a = \sum a_i$ over these parts only. All are unqualified destinations. The ordered inequality in §1 supplies at least $2\sum \mu_i \ge 2(r-1)Or$ outgoing target edges from them. Count just the incidence at the earlier endpoint of each such edge. These incidences are distinct, and every one is absent from Y.

To account for overlap, distribute exactly $r/4$ units of edge weight over each first-stage recorded star and exactly τr units over the fresh edges at each second-stage source, with at most one unit per edge. At the first stage count this weight at both ends, and at the second count it only at the source. These are the charges already used for Y_*. An incidence in the preceding outgoing count cannot be a first-stage destination charge, since its part is unqualified. All its possible overlap is therefore at an early source in these parts. The total overlap is at most $\tau a r^2$, including sources from both stages. It follows that

$$Y \le Y_* - 2(1 - 1/r)O + \tau a.$$

Here we discarded the favorable $2v$. The sum on the right of (2) over chosen parts is at most $O - a + q/256$. The table parameters satisfy $2(1 - 1/r)\beta \ge \alpha$ and $\tau\beta \le \alpha$. Combining the last two bounds pays for both O and a. Since $q \le n/2$, we obtain

$$\frac{Y}{\alpha} + \frac{D}{\beta} \le \frac{Y_*}{\alpha} + \frac{p + t}{\beta} - \frac{d - 1/256}{2\beta}\, n. \tag{3}$$

3. Endpoint bounds for the width integral

Take $K = 1120,\ K' = K + 2n$. Give each ordinary row weight $a_i = \max(y'_i, \alpha x_i/(2\beta))$, reusing a_i only in this section. Set aside the $K' - e_0 - 1$ largest weights. There are at least $K'/4$ such rows, because $e_0 + 1 \le 729 + n$ and $729 \le 3K/4$. Put $Q_i = \alpha x_i^2/(4\beta^2) \le a_i^2/\alpha$. For the rows still under consideration define

$$s(z) = K' + \#\{i : a_i > z\}, \qquad F(z) = s(z)z/\alpha + \sum_{a_i \le z} Q_i/z.$$

Let $R = 2^b$ with b from the table in §5, $S = 2^{(146-b)/4}$, $\ell = \alpha/S$, and $H_0 = \alpha/K'$. We have $K' \le 3942$ and $S > 2K'$. The full integral of a positive-weight row against dz/z is $a_i/\alpha + Q_i/a_i \le y'_i/\alpha + x_i/\beta$. A zero-weight row costs zero. The contribution of the $K'z/\alpha$ term on $[0, H_0]$ is one.

Suppose $F \ge 1$ on $[\ell, H_0]$. Its integral below ℓ is at least $F(\ell) \ge 1$. For $a_i > \ell$ the integral there equals the row's value ℓ/α. For $0 < a_i \le \ell$ it is $a_i/\alpha + Q_i/a_i - Q_i/\ell \ge Q_i/\ell$, since $a_i/\alpha \ge Q_i/a_i \ge Q_i/\ell$. The base term also equals its endpoint value after integration.

There is a further saving of $1/4$ from the rows set aside and the square terms above H_0. Write $u = \max a_i/H_0$ over the remaining rows, taking zero on the empty family. The rows set aside cost at least $u/4$ in total. If $u \ge 1$, this suffices. If $u \le 1/2$, failure at $H_0/2$, where every remaining row has passed its threshold, gives $\sum Q_i/H_0 \ge 1/4$. This is precisely their integral above H_0. For $1/2 \le u \le 1$, failure at uH_0 gives $\sum Q_i/H_0 \ge u(1-u)$. Together the two savings are at least $u/4 + u(1-u) \ge 1/4$, since the difference is $(1-u)(u-1/4)$.

Failure throughout the interval would imply $\log(S/K') \le Y/\alpha + D/\beta - 1/4$. By $\log(K'/K) \le 2n/K$ and $(d - 1/256)/(2\beta) \ge 2/K$, (3) would give $\log(S/K) \le Y_*/\alpha + (p+t)/\beta - 1/4$. Section 5 verifies the strict reverse. Take a level with $F(z) < 1$, set $s = s(z)$ and $w = sz/\alpha$, and omit the remaining rows of weight above z. There are exactly $h = s - 1$ omitted slots. On the retained cliques,

$$s \max y_i' \le \alpha w, \quad s \max x_i \le 2\beta w, \quad s \sum x_i^2 \le 4\beta^2 w(1-w), \quad K \le s < S, \quad r \ge Rs^4. \tag{4}$$

The first two inequalities follow from the threshold. The third follows by multiplying the inequality $\sum Q_i/z < 1 - w$ by $4\beta^2 sz/\alpha$.

4. A shortage of at least $3r/20$

Use $f = 9/40$, $\rho = f/s$, $\rho_0 = f/K$. Let G consist of all exterior points with at most ρr neighbors in each retained clique, and put $\Delta = hr - |G|$. We show $\Delta > cr$ for $c = 3/20$. Let A, X_0, B upper bound the first three left sides in (4). Ordinary lists of size at least $r/41$ in any subfamily have independent representatives by the counting lemma, since inter-clique edges are undiscounted and $4 \cdot 41^2 A/K < 1$. Independent $h+1$ points of G would therefore extend through all ordinary rows. In a temporary ordinary row remove vertices whose **unrecorded** foreign degree exceeds $r/8$. Its remaining fraction is at least $1 - o$, $o = 1/256$, since $(X_0 + 8A)/K < o$. The degree from a survivor outside its original clique is at most $(1/8 + \omega + 1/256 + X_0/K)r < (7/50)r$. This includes the points in its own raw part outside the clique. Ordinary list counting lemma

If $\Delta \le cr$, take $hr + 1$ points of G when available and otherwise take G. This auxiliary set has order $hr + b'$ for $b' = 1$ or $b' = -\Delta$, $0 \le \Delta \le cr$ in the second case. Its target independence number is at most h. Write $I = 2^{-20}$, so $h \ge 799$, $h^2/r < I$, $1/r < I$. The target and the $r - 1$ other Turán floors give $2\eta_a/r^2 \le \xi = c^2 + I$. In fact $2\eta_a \le rh(h-1) + c^2 r^2$ by the signed-integer floor expansion in base §3. For the h auxiliary parts of (1), put $x = X_a/r^2$, $u_i = (r - |R_i^a|)_+/r$. Since the balanced squared deviation is at most $(b')^2/h + h/4$, (1) yields

$$x \le \xi, \qquad x + \sum u_i^2 \le T = (1 + 1/799)c^2 + 3I, \qquad \sum u_i \le D_a = c + \frac{\xi}{2(1-I)}.$$

The last bound charges each overfull auxiliary part to the other colors and adds the net deficit. Turán floor and auxiliary construction

Prune auxiliary vertices of current interpart degree above $t_a r$, $t_a = 43/200$. If zr vertices are pruned, $0 \le z \le x/t_a$ and at most $(x - t_a z)r^2$ interpart edges remain. For any $j = 1, 2, 3$ rows the sum of their final positive deficits, divided by r, is at most

$$D_j = b_j + (T - b_j^2/j)/t_a, \qquad b_j = \min(D_a, jt_a/2), \qquad D_0 = 0.$$

Indeed their raw sum u satisfies $x \le T - u^2/j$, and all subsequent deletions cost at most x/t_a in total. The quadratic maximum on $[0, D_a]$ is the value displayed. In particular $u_i + x/t_a \le D_1$. The three bounds are below 0.158546, 0.205577, 0.225734. Every auxiliary survivor part is a clique. A nonedge would leave

fractions at least $1 - D_1 - 2t_a > 0$ in the other auxiliary parts. Its square exceeds ξ, so the expected number of interpart edges among uniform representatives is less than one.

Use thresholds $\theta_a = 81/200$ and $\theta_o = 369/1000$ for large attachments to auxiliary and ordinary rows respectively. As in the earlier attachment proof, extend either two independent exterior points or one exterior point and at most two compatible prescribed representatives. Extra rows may be omitted. Each unfilled row has only small attachments from the prescribed exterior points. To fill auxiliary rows, it suffices that each list be nonempty and the product of any two list fractions exceed the entire normalized remaining edge count $x - t_a z$. Uniform independent choices then have expected conflicts less than one.

For two exterior points let $w_0 = 1 - 2\theta_a$, and sort the raw lower bounds $b'_1 = w_0 - u_1,\ b'_2 = w_0 - u_2$ so $b'_1 \le b'_2$. If $a = u_1 + u_2$, their product plus $u_1^2 + u_2^2$ is at least $w_0^2 - w_0 a + 3a^2/4$. Its minimum for $0 \le a \le D_a$ exceeds T, so $b'_1 b'_2 > x$. Both bounds are positive since $w_0 > D_1$. Charging all z deletions against the smaller factor leaves product at least $b'_2(b'_1 - z)$. This exceeds x at zero and is positive at $z = x/t_a$, since $u_i + x/t_a \le D_1$. Linearity yields the required strict inequality.

For one exterior point use $w_0 = 1 - \theta_a$. At most two representatives forbid at most $2t_a r$ points across all unfilled auxiliary rows. Ordinary representatives have the smaller outside-clique degree bound $7r/50$. Here $w_0 - 2t_a > D_1,\ w_0 - D_a > t_a$. Thus $b'_1 - 2t_a > x/t_a,\ b'_2 > t_a$, giving $b'_2(b'_1 - 2t_a - z) > x - t_a z$. In both cases the same individual estimates give nonempty lists even if only one row is unfilled. All auxiliary choices are in G. Their total cost in any ordinary row is at most fr. The available fractions $1 - o - 2\theta_o - f$ and $1 - o - \theta_o - f - 2(7/50)$ both exceed $1/41$.

For the $hr + 1$ auxiliary set use the h auxiliary rows alone. Otherwise use them with the $r - h$ temporary ordinary rows on the full graph. In each case the vertex count is one more than the number of rows times r, and the target independence cap is the number of rows. Every exterior point has a large attachment. A point with a unique large row must join all its vertices, and any two such points assigned to that row are adjacent, by the preceding extensions. Enlarge the cliques by those points. If P exterior points remain, the enlarged deficits in vertices sum to $P - 1$.

For completeness, price each large arrival at $(kd_i + q_0)r$, where d_i is the enlarged normalized deficit, $k = 77/100,\ q_0 = 1/10000$. Write ϑ for its row threshold and e for its individual deficit cap o or D_1. The checker tests

$$2(\vartheta - ke - q_0)(1 + k) > 1, \quad 2(\vartheta - ke - q_0)q_0 > (\vartheta + e + 1/2)I, \quad 2((1 + k)e + q_0) < 1.$$

The one-clique capacity applied to $\lfloor 2((1 + k)d_i + q_0)r \rfloor$ arrivals then gives $(n + 1)/(2r) - d_i \le kd_i + q_0$ for the actual count $n > 0$, including rounding as in §3 of Appendix C.

For a point with two large rows let j be the number of auxiliary ones, let e_1, e_2 be the individual caps before enlargement, and set $U = D_j + (2 - j)o$. If their combined degrees were at most $(1 + kU + 2q_0)r$, the two lists of nonneighbors would have product larger than the entire target cut between these cliques. For each ordered pair of types the checker verifies positive factors and

$$(1 - e_1 - \vartheta_1)(\vartheta_1 - e_2 - kU - 2q_0) > \min(e_1 e_2, U^2/4) + I.$$

The minimum product at fixed combined degree is at an endpoint by concavity. The two-clique capacity bounds the normalized cut by the right side. Compatible representatives would extend with the point. Its net contribution after prices must therefore exceed r. Any three large arrivals also pay more than r net, since $j\theta_a + (3 - j)\theta_o - k(D_j + (3 - j)o) - 3q_0 > 1$ for $0 \le j \le 3$. Every additional arrival contributes positively. Summing the one-clique capacities gives $rP \le (r - 1)(P - 1)$, impossible. For $P = 0$ the sum of deficits is already impossible. This proves the claimed shortage. Attachment prices and rounding

5. The five width bands and the final signed cut

In the following table $b = \log_2 R$. Select a closed band containing Y_*.

Band for Y_*	b	4000α	4000β
0 to 0.50	14	656	924
0.50 to 0.55	14	656	922
0.55 to 0.68	14	652	916
0.68 to 1	15	652	916
1 to 2	17	654	918

For each band the checker clips the pruning trapezoid by its two boundary lines. At every vertex $Y_*/\alpha + (p+t)/\beta - 1/4 < \log(S/K)$. The minimum certified difference exceeds 0.01427. Logarithms are enclosed by the positive series for $2\,\mathrm{arctanh}$.

The first band also identifies a limit of these constants. At the relaxed boundary $p = 125/61,\ Z = 0$, we have $Y_* = 119/244,\ t = 11900/12139$. For $r = 2^{145}$, the sufficient width cost there exceeds $\log(S/K)$ by more than 0.159 when $S = 2^{(145-14)/4}$. Whether actual edge incidences can consume that entire budget is a separate question.

Apply the marked-sender and excess-arc construction of §5 of Appendix B. The per-destination degree bound is $r/4$ between retained cliques. A retained row has at most ωr recorded sources, and $D+1/2+p < 7,\ 2D < 9$. The ordinary extension for excess arcs requires $1 - X_0/K - 3/4 - f > 1/41$, which holds. Change the unique-sender retry probability to $100\rho < 1$. The success probability at neighbor fraction $q_y > 1/2$ is $(1-q_y)(1+100\rho q_y)$. The marginal probability of a given representative is at most j/r for

$$\gamma = A/f, \quad g = X_0/f, \quad \kappa = B/f, \quad j = (1+100\rho_0)/(1-g\rho_0).$$

The retry surplus is at least $37\rho r(1-q_y)$ by $50(1-\rho_0) - 1 - g > 37$. The excess-arc conflict coefficient stays $4/3$ and the variance coefficient stays $\lambda = 501/1000$, since $2g\rho_0/(1-g\rho_0) < 1/1000$. Excess arcs and tested cut · Retry calculation

Here are the remaining inputs to that signed cut. Put $\eta' = 1/(Rf^2)$, y_{hi} equal to the upper end of the band, and

$$\begin{aligned} n_* &= 513\gamma/512 + 1/256 + g\rho_0 + \omega, & a &= \min(5/2, (249/250)/n_*), \\ k' &= (4/3)/(a-4/3), & b_1 &= j(1+1/k'), \\ V' &= a(513y_{\mathrm{hi}}/(512f) + 7/K) + 9b_1/K. \end{aligned}$$

The heavy-load count at each row is at most $n_* r$, using $2jg\rho_0 < 1/512$. The tests $an_* + 2b_1 g\rho_0 < 1,\ a > 4/3,\ a\kappa < 2$ give valid row probabilities and the square estimate of Appendix B. Its repair bound in units of $\rho^2 r^2$ is

$$P(W', l) = \frac{k'(b_1 W' + l)^2}{2 - a\kappa} + k'\eta' \left(\frac{(1+V')^2 + V'/K}{2} + \frac{1}{2K} \right).$$

Here $l = (\Delta - m)/(\rho r)$, m is the expected number of marks, and $W'\rho r$ bounds the sum of unique-sender counts times their assigned normalized deficits. The first application uses 2κ. The retry surplus gives the second input

$$Z_1 = 8P(2\kappa, 0) + 2\eta', \qquad W_1 = ((1-g\rho_0)^{-1} + \lambda g/37)\kappa + gZ_1\rho_0/37.$$

Put $Z_2 = 8P(W_1, 0) + 2\eta'$ and $\epsilon(l) = P(W_1, l) + gl + l^2/2 + \eta'(1/2 + 1/(2K))$. Both oversampling tests $Z_i\rho_0^2 < \min(c/128, 1/16)$ pass. The one-row signed-cut comparison and the bound on expected failures require $0 \le l \le \lambda\kappa$ and

$$(1-\epsilon(l))(c - \lambda\kappa\rho_0) + l - \lambda\kappa - Z_2\rho_0 - 1/(fRK^2) \le 0. \tag{5}$$

These reductions use the shortage from §4 and the same surplus coefficient 37. Two applications of the signed cut

Divide $w \in [0,1]$ in (4) into 128 closed intervals per band. Use the maximum of each of its first three right sides separately on each interval for A, X_0, B. The checker uses these upper bounds throughout, verifies $0 < \epsilon(l) < 1$ at both endpoints of the l interval, and verifies all the list, degree and probability conditions above. The left side of (5) is concave in l, since $c - \lambda\kappa\rho_0 > 0$. All $5 \cdot 128 \cdot 2 = 1280$ endpoint values exceed 0.0003327, contradicting (5). This proves the theorem subject to the cited earlier arguments. The exact computation certifies the constants and tests the discarded-incidence bound on all 1,024 labelled five-vertex target graphs with every choice of omitted raw parts. Five local inequalities are checked in Lean. The general combinatorial argument is in the manuscript.

Appendix B. Excess loads and the signed cut

Sections 1 and 2 analyze the original choice of rows. Sections 3 to 5 give the general estimates used in Appendix A.

1. Destinations of the recorded stars

Assume a counterexample. Fix its sparsest color J. Every color has independence number at most r. In the stability partition R_i, put $b_i = |R_i| - r$, and let μ_i be the number of internal pairs missing J. Write X for the number of interpart edges of J, and η for its excess over the complementary Turán floor $p_r(r^2+1)$. The base manuscript gives $\mu = \sum \mu_i \le \eta \le \binom{r}{2}$, $\sum b_i = 1$, $\sum (b_i)_+ \le r/2$, and $X = \eta + \mu - (\sum b_i^2 - 1)/2$. A destination qualifies when $b_i \le r/256$ and $U_i = \mu_i - (r-1)b_i \le r^2/1200$. At most 728 parts fail. At most ωr exterior vertices originally have more than $r/4$ target neighbors in any given qualifying part, where $\omega = 5/1464$. These are the estimates of §1 of Appendix D.

Keep that extraction unchanged. A first-stage removal has more than $r/4$ current edges to a qualifying foreign part. A second-stage removal has current total interpart degree above τr, with $\tau = 47/100$. Put pr, tr for the numbers removed and $Z = 1 - p/4 - \tau t$. Trim parts to r, select the original maximal family of q jointly target-independent pairs in distinct parts, and delete their neighbors in unchosen parts. Ordinary cliques C_i come from qualifying unchosen lists of size at least Lr, $L = 19/20$. Write $x_i = 1 - |C_i|/r$, $y_i = e_J(C_i, V \setminus R_i)/r^2$ and $D = \sum x_i$.

Record one destination of each first-stage removal. A source recorded at an ordinary destination i has more than $(1/4 - x_i - b_i/r)r$ edges into its **final** clique, since initially $|R_i \setminus C_i| = x_i r + b_i$. These are foreign edges counted by y_i. Distinct sources at the destination give different edges. For all $r \ge m$ the number of sources recorded at ordinary destinations, divided by r, is thus at most

$$P = \sum_i \min\left(\omega, \frac{y_i}{1/4 - 1/256 - x_i}\right) < 609/1000.$$

For $j = i + 3 = 4, \dots, 255$ the checker sums the rational bounds using $x_i \le 1/(5j) + 1/m$, $y_i \le 1/(100j) + 1/m^2$. For $j \ge 256$ use the common denominator $1/4 - 1/256 - 1/1280 - 1/m$. The sum of the numerators is at most $\log((m+3)/255)/100 + 1/m$. No positive lower bound on a foreign edge count is needed.

Let F be the number of short unchosen lists and $W = \sum_{\text{short}} (1 - |C_i^{\text{list}}|/(Lr))$, using all unchosen lists at this point. A qualifying recorded destination omitted from the ordinary rows is chosen or short. Therefore

$$p \le P + \omega(q + F), \qquad q \le W + Z/L^2, \qquad W \le F.$$

The middle inequality is the second representative draw in §1 of Appendix D. The deficit of each short list in units of r is exactly $1 - L + L(1 - |C_i^{\text{list}}|/(Lr))$. Subtracting these deficits from $1/2 + p + t + 2\tau q$ pays for their omission. With $c = 1 - 1/(4\tau)$, $C_0 = 1/2 + 1/\tau + c(609/1000)$, and $e = 1/\tau - 2\tau/L^2$, this gives

$$D \le C_0 + c\omega q - (1 - L - c\omega)F - (L - 2\tau)W - eZ \le C_0 - (1 - 2\tau - 2c\omega)q. \tag{1}$$

For the last step, $1 - L - c\omega > 0$, $L > 2\tau$, and $e > (1 - 2\tau - c\omega)/L^2$. The certificate in §2 gives $D > 8/3$ throughout the feasible range. Equation (1) implies $q \le 4$, then $F \le 5$ by its first inequality, and consequently $p < 16/25$.

2. Pair neighborhoods in ordinary parts

Let S consist of the first- and second-stage sources in the original parts that eventually yield ordinary rows. Write $a_i r = |S \cap R_i|$ there and $p_R r, t_R r$ for the two removal counts in S. Let $B_R r = \sum_{\text{ordinary}} (b_i - a_i r)_+$ count subsequent trims there and $O_E r = \sum_{\text{omitted}} (b_i)_+$. Write U for the number of vertices removed from ordinary parts by the pair neighborhoods alone, in units of r.

The actual target edges lost when deleting S include at least $(p_R/4 + \tau t_R)r^2$ fresh interpart edges spent by its sources. They also include the same-part edges from $S \cap R_i$ to C_i. The number of missing pairs on those cuts is at most

$$\sum_{\text{ordinary}} \mu_i - (r-1)B_R r.$$

Indeed, the raw parts remaining after the first two stages still contain at least $(r-1)(b_i - a_i r)_+$ nontarget pairs. Each of the other colors needs at least $(b_i - a_i r)_+$ internal edges by its independence cap. These pairs avoid S. In the omitted raw parts the same argument before pruning reserves at least $(r-1)O_E r$ missing pairs.

On the other hand, deleting S loses at most $r|S| + \eta$ target edges. The surviving graph has independence at most r, and each decrement of $p_r(n)$ for $n \le r^2 + 1$ is at most r. Comparing both estimates and using $\eta + \mu \le r^2$ yields

$$p_R/4 + \tau t_R \le 1 - (1 - 1/r)(B_R + O_E) + \sum a_i x_i.$$

The exact count of vacancies in ordinary parts gives $D = p_R + t_R + B_R - \sum_{\text{ordinary}} b_i/r + U$. Here $-\sum_{\text{ordinary}} b_i/r \le O_E$. Since $1 - (1-1/r)/\tau < 0$, elimination of t_R gives

$$D \le 1/\tau + c p_R + \frac{1}{\tau}\sum a_i x_i + U. \tag{2}$$

Each $a_i \le x_i + (b_i)_+/r \le x_i + 1/256$. The ceilings give $\sum x_i^2 < 14/1000$, by $\sum_{j=4}^{\infty} j^{-2} \le 1/3$ and $\sum_{j=4}^{m+3} 1/j < 16$. Use $p_R \le p < 16/25$, $D \le C_0$ inside the error term, and $D > 8/3$ on the left of (2). We get $U > 9/50$. Even the five largest allowed deficits sum to less than $9/50$. Thus at least six ordinary parts lose neighbors and $q \ge 1$. Every chosen endpoint misses every final ordinary clique in color J.

For clarity, the arithmetic uses the reduction in the next table. Put $T(r) = \sum_{j=4}^{m+3} \lfloor r/(5j) \rfloor$, so that $D \ge T(r)/r$.

Range of r, left endpoint included	Certified lower bound on $T(r)/r$	Possible q from (1) and (2)
2^{24} to 19,000,000	8/3	1 through 4
19,000,000 to 25,000,000	2.690	1 through 3
25,000,000 to 33,000,000	2.745	1 or 2
33,000,000 to 40,000,000	2.800	1
40,000,000 to 43,000,000	2.840	1
43,000,000 and above	2.856	none

In the two upper surviving bands, substituting 2.800 and 2.840 on the left of (2) requires the single pair's neighbors to meet at least fourteen and eighteen ordinary parts respectively, by summing the largest allowed deficits. Above forty million, including the endpoint, (1) also gives exactly one short list. Two short lists would give $D < 2.840$; zero would require $Z \ge L^2$, also impossible there. With $q = F = 1$, (1) gives $Z/L < 13/500$. The unique short list has length $(1 - W)Lr \le Zr/L < 13r/500$. The other ordinary parts keep their original clique memberships throughout this argument.

For each integer interval $[a, b]$ in the certificate, $T(r)/r \ge T(a)/b$. The checker calculates $T(a)$ by grouping identical integer quotients. The blocks cover every integer through $8m - 1$. For $r \ge 8m$, $T(r)/r \ge (\sum_{j=4}^{255} 1/j + 16 \log 2)/5 - 1/8 > 2.856$. This proves Theorem 2 without a monotonicity assumption on the normalized sum. Floor certificate and fixed-row inequalities

3. Pair selection before trimming

We may now choose rows for an arbitrary counterexample. Use the same first-stage pruning of the sparsest color. Write pr for the count and keep the recorded stars of §1 of Appendix C. Prune next at total interpart degree τr, now $\tau = 43/100$, with tr removals. Write $Z = 1 - p/4 - \tau t$, $V_2 = \sum (b_i/r)^2$ and $Y_* = 2 - p/2 - \tau t$. The actual remaining interpart count in units of r^2 is at most $z = Z - V_2/2$ by the stability identity. Choose a maximal family of q jointly independent pairs in distinct surviving parts **before trimming any part**. Remove their neighbors from the unchosen parts. Each resulting list is a clique by maximality and already has size at most r.

Put $L = 124/125$ and $O_Q = \sum_{\text{chosen}} (b_i)_+/r$. The total deficit across unchosen lists in units of r is at most $p + t + O_Q + 2\tau q$. Indeed deleting a whole chosen *surviving* part costs at most $r + (b_i)_+$, the original order is $r^2 + 1$, and the pair endpoints have remaining interpart degree at most τr. Use the two padded draws from Appendix D. The first gives $q \le (1/2 + p + t + Z)/(1 - 2\tau)$. In the second, $q \le W + z/L^2$, with W defined as in §1. Omit chosen, short and unqualified parts. For ordinary deficits $D = \sum x_i$ and discounted foreign counts $Y = \sum y_i'$ of Appendix C we obtain

$$D \le p + t + O_Q - (1 - 2\tau)q + z/L^2, \qquad Y \le Y_* - V_2. \tag{3}$$

Indeed each short list pays at least its contribution to W. Each first-stage recorded edge is absent from the discounted incidence count at both ends and each edge spent by a second-stage removal is absent at at least one end. The original interpart incidence budget is at most $(2 - V_2)r^2$. The per-destination degree bound $r/4$ still holds for surviving points in other retained cliques.

For width parameters α, β from §5 put $g_0 = 1/(2L^2) + \beta/\alpha$. Cauchy gives $O_Q^2 \le qV_2$, including $q = 0$. Since $4(1 - 2\tau)g_0 \ge 1$, the arithmetic-geometric mean inequality gives $O_Q \le (1 - 2\tau)q + g_0 V_2$. Hence

$$\frac{Y}{\alpha} + \frac{D}{\beta} \le \frac{Y_*}{\alpha} + \frac{p + t + Z/L^2}{\beta}. \tag{4}$$

For a separate unweighted bound, each short list pays its 2τ multiple in the second draw with at least $\min(1 - L, 1 - 2\tau)$ left. Thus $D \le D_* = 1/2 + p + t + 2\tau Z/L^2 < 3.7$, and the number of short lists is at most $D_*/\min(1 - L, 1 - 2\tau)$. With the chosen and unqualified parts this gives $e_0 + 1 < K/2$ for the total omission count e_0 and $K = 2560$. The four corners of the trapezoid $0 \le p \le 125/61,\ t \ge 0,\ Z \ge 0$ suffice for both checks.

4. Omitted rows at minimum width

Give each ordinary row weight $a_i = \max(y_i', \alpha x_i/(2\beta))$. Set aside the $K - e_0 - 1$ heaviest ordinary rows. For the rest put $s(z) = K + \#\{i : a_i > z\}$ and $C(z) = \sum_{a_i \le z} x_i^2/z$. If $s(z)z/\alpha + \alpha C(z)/(4\beta^2) < 1$, the width proof of Appendix D gives $h = s(z) - 1$ omitted slots and $v = s(z)z/\alpha$, with

$$s \max y_i' \le \alpha v, \qquad s \max x_i \le 2\beta v, \qquad s \sum x_i^2 \le 4\beta^2 v(1 - v). \tag{5}$$

Here $s = s(z)$. For $R = 2^b$ set $S = 2^{(165-b)/4} > 2K$ and integrate the left side of the test against dz/z from α/S to α/K. The constant K costs less than one. Each row's full cost, integrating its two terms on the positive line, is $a_i/\alpha + \alpha x_i^2/(4\beta^2 a_i) \le y_i'/\alpha + x_i/\beta$. A zero-weight row costs zero.

There is a saving of at least $1/4$ unless (5) already holds at $z_0 = \alpha/(2K)$. If any remaining weight exceeds z_0, each of the rows set aside costs at least $1/(2K)$ in the full bound. There are at least $K/2$ such rows. Otherwise $s(z_0) = K$; failure of the test there implies $K \sum x_i^2 \ge \beta^2$ on the remaining rows. The omitted part of their square integral above α/K then costs at least $K \sum x_i^2/(4\beta^2) \ge 1/4$. Thus a sufficient

condition for (5) at some $K \le s < S$ is

$$\frac{3}{4} + \frac{Y_*}{\alpha} + \frac{p + t + Z/L^2}{\beta} < \log(S/K).$$

It follows that $r \ge Rs^4$ throughout the claimed range.

5. The excess above the load threshold

Use the actual target edges, exterior set G, unique-sender trials and marked vertices of Appendix C. Set $f = 7/30,\ \rho = f/s,\ \rho_0 = f/K$ and $c = 1/8$. Its auxiliary argument gives $\Delta = hr - |G| > cr$ under (5) and the list checks below. The new extraction has the same per-destination degree and recorded-star bounds needed there. The untrimmed deficit bound gives $\sum x_i + 1/2 + p < 7$ and $2\sum x_i < 9$, as needed in its estimates of sender counts.

In an unrepresented row the original load t_{xi} of a marked vertex x is its own target degree into that row plus the degree of its representative there, if present. It is zero in its represented row or when unmarked, and at most $3r/4$. Place an arc of length $(t_{xi} - \rho r)_+$, for positive lengths only, on a circle of circumference $3r/4$, with independent uniform starts. In an independent set of $h + 1$ exterior points of the repaired graph, the excess arcs in an unrepresented row are disjoint and have total length at most $3r/4$. The baseline loads of all these points cost at most $(h+1)\rho r = fr$, including the points of G. Exactly the same ordinary lists can therefore be filled as in the 190 proof. Real half-open arcs allow nonintegral lengths.

Write E_i for the expected true cut from the sampled exterior set to row i, F_i for expected loads from representatives in other rows, n_i for the expected number of heavy loads and N_i for the number of unique strong senders. With the notation of §4 of Appendix D set

$$Q = \sum n_i(E_i + F_i)/r, \quad H = \rho \sum n_i^2, \quad T = \sum N_i E_i/r, \quad H_E = j^2 \sum_{i<k} N_i N_k e_J(C_i, C_k)/r^2.$$

Two arcs intersect with probability at most the sum of their lengths divided by the circumference. In the ordered-pair sum, each sender's heavy indicator is independent of each distinct sender's excess. The total expected excess at row i is at most $E_i + F_i - \rho r n_i$. Extending the sum of the nonnegative products to all indices therefore bounds the arc conflicts by $4(Q - H)/3$. The representative conflicts from the 218 proof are unchanged. The repair bound is now

$$\mathcal{R} = \tfrac{4}{3}(Q - H) + jT + H_E.$$

This bound scales by at most four when the thinning probability increases from p_0 to $p_1 \le 2p_0$. Indeed n_i is linear in the probability. The other factor $E_i + F_i - \rho r n_i$ is a nonnegative constant from G plus the probability times a nonnegative coefficient, since each marked trial's counted incidences dominate ρr times its heavy indicator. Thus $Q - H$ scales by at most four, T by two and H_E stays fixed.

For bounds A, X_0, B on the three left sides of (5), put $\gamma = A/f,\ g = X_0/f,\ \kappa = B/f$, $j = (1 + 80\rho_0)/(1 - g\rho_0)$, and $\eta_0 = 1/(Rf^2)$. Appendix C gives

$$n_i/r \le z_* = 513\gamma/512 + 1/256 + g\rho_0 + \omega, \qquad \sum n_i \le M_0 sr, \quad M_0 = 513Y_{\mathrm{hi}}/(512f) + 7/K.$$

Use a band containing Y_* and write Y_{hi} for its upper end. In the tested-row inequality use

$$a = \min(5/2, (249/250)/z_*), \quad k = (4/3)/(a - 4/3), \quad b_1 = j(1 + 1/k), \quad V = aM_0 + 9b_1/K.$$

All tested probabilities $(an_i + b_1 N_i)/r$ are at most one. Suppose $\sum N_i d_i/r \le W\rho r$, where $d_i = x_i r$, and put $l = (\Delta - m_0)/(\rho r)$ for expected mark count m_0. By Cauchy, $\sum n_i d_i/r \le \sqrt{\kappa H}$. The tested cut pays $aQ + b_1 T + (b_1/j)^2 H_E$ in actual edges as in (5)–(6) of Appendix D. After deducting repairs, it leaves

$(a-4/3)(Q-H)+aH$, together with $(b_1-j)T+((b_1/j)^2-1)H_E$, on the left. On the right the squared term is at most $(a\sqrt{\kappa H}+(b_1W+l)\rho r)^2/2$. For $a\kappa<2$, completing the square gives

$$\tfrac{1}{2}(a\sqrt{\kappa}\,x+z)^2-ax^2\le z^2/(2-a\kappa).$$

Multiplication by k now pays the three repair terms. The remaining height and mean terms from the tested cut give $\mathcal{R}\le P(W,l)\rho^2r^2$, where

$$P(W,l)=\frac{k(b_1W+l)^2}{2-a\kappa}+k\eta_0\left(\frac{(1+V)^2+V/K}{2}+\frac{1}{2K}\right).$$

All other steps in the two applications of the signed cut in §4 of Appendix C use only this repair bound, its oversampling version and the unchanged one-row capacities.

The table uses $K=2560$ throughout; b is the exponent of R.

Band for Y_*	b	4000α	4000β
0 to 5/8	14	676	880
5/8 to 3/4	15	680	884
3/4 to 7/8	15	680	880
7/8 to 1	15	676	878
1 to 9/8	16	680	885
9/8 to 5/4	16	680	880
5/4 to 11/8	16	680	880
11/8 to 13/8	16	676	872
13/8 to 2	17	680	880

The width cost is affine on each clipped trapezoid, so its vertices suffice. In each band divide $v\in[0,1]$ into 96 equal intervals, bounding the three quantities in (5) separately by their maxima on each interval. For completeness write $\lambda=501/1000$, $Z_1=8P(2\kappa,0)+2\eta_0$, and use Appendix C's retry estimate

$$W_1=\left((1-g\rho_0)^{-1}+\lambda g/37\right)\kappa+gZ_1\rho_0/37.$$

Put $Z_2=8P(W_1,0)+2\eta_0$, and $\epsilon(l)=P(W_1,l)+gl+l^2/2+\eta_0(1/2+1/(2K))$. At each endpoint of $0\le l\le\lambda\kappa$, the checker gives

$$(1-\epsilon(l))(c-\lambda\kappa\rho_0)+l-\lambda\kappa-Z_2\rho_0-1/(fRK^2)>1/2000.$$

This expression is concave in l. The one-row and oversampling inequality requires it to be nonpositive. All 1,728 endpoints pass, as do the ordinary and auxiliary list conditions, the two oversampling caps, and the height bounds. Together with the earlier sampling bounds, this proves Theorem 1. The Lean proof guide gives the formal argument for the sharper cutoff. Arithmetic and arc checks · Combined manuscript

Appendix C. Auxiliary rows and repeated choices

1. Distinct sources of large stars

Assume a counterexample. Fix a sparsest color J. Every color has independence number at most r and every monochromatic clique has size at most r. Use the partition R_i and qualifying destinations from §1 of Appendix D, including its first pruning stage at $r/4$. For each first-stage deletion record its source, one qualifying destination, and the *current* edges from that source into that destination. All recorded edge sets are disjoint. There are pr sources. The preceding star bound gives at most

$$\omega r = \frac{5}{1464} r$$

recorded sources per qualifying destination. Indeed its budget $U_i \le r^2/1200$ pays more than $(61/250)r$ per source. The upper bound on p is $p_* = 125/61$.

Use total interpart degree threshold τr, $\tau = 99/200$, in the second pruning stage, with tr deletions. Put $Z = 1 - p/4 - \tau t \ge 0$. Trim oversized parts to r, take a maximal family of q jointly independent pairs in different parts, one pair per chosen part, and remove their neighbors in the unchosen parts. Each remaining list in an unchosen part is a clique. Use the two independent representative draws in Appendix D with short-list cutoff Lr, $L = 497/500$. Omit chosen parts, short lists and unqualified destinations.

Write $d_i = r - |C_i|$, $x_i = d_i/r$. Let $y_i' r^2$ count the edges of J from C_i outside R_i after excluding the recorded edges. This discount is used only for bounding degrees and numbers of heavy senders. Write $b_i = |R_i| - r$. The budgets are

$$\sum x_i \le D_* = \tfrac{1}{2} + p + t + 2\tau Z/L^2, \qquad \sum y_i' \le Y_* = 2 - p/2 - \tau t. \tag{1}$$

Indeed the original interpart edges have at most $2r^2$ endpoint incidences. Each fresh recorded edge loses **both** incidences in this count. Its source was deleted, and its other endpoint was either removed or discounted. Every fresh edge spent in the second stage loses at least one incidence. Inter-clique edges lose neither incidence through the discount.

For the first budget the padded draws give $q \le (1/2 + p + t + Z)/(1 - 2\tau)$. In a short list of length zr, $0 \le z < L$, the excess after paying $2\tau(1 - z/L)r$ is at least $r \min(1 - 2\tau, 1 - L)$, by linearity in z. Thus the number of short lists is at most $D_*/\min(1 - 2\tau, 1 - L)$. On the trapezoid $0 \le p \le p_*$, $t \ge 0$, $Z \ge 0$, the four corners give $D_* < 18/5$ and fewer than 2399 omitted parts, including the at most 728 unqualified ones. Let e_0 be the actual omission count.

For the width, use the threshold proof of Appendix D with y_i' replacing y_i. Give row i weight $\max(y_i', \alpha x_i/(2\beta))$. With $s(z) = \max(K, e_0 + 1 + \#\{\text{weights} > z\})$, integrate as in its §2. The integral costs at most $1 + Y_*/\alpha + D_*/\beta$. In the following table α, β are displayed in thousandths. Choose a band containing Y_*.

Band for Y_*	K	b	1000α	1000β
$[0, 5/8]$	2400	15	155	200
$[5/8, 7/8]$	2600	16	165	198
$[7/8, 9/8]$	2600	18	170	200
$[9/8, 11/8]$	2600	18	170	198
$[11/8, 13/8]$	2600	18	170	198
$[13/8, 2]$	2600	19	170	198

Set $R = 2^b$, $S = 2^{(190-b)/4}$. In each band the integral bound is strictly below $\log(S/K)$. Its maximum occurs at a vertex of the clipped trapezoid. The checker bounds the logarithms by the positive series for

2 arctanh. As in the 218 proof, there is a width $K \le s < S$ and $0 < v < 1$ such that on the $r-h$ retained rows, $h = s-1$,

$$s \max y_i' \le \alpha v, \qquad s \max x_i \le 2\beta v, \qquad s \sum x_i^2 \le 4\beta^2 v(1-v), \qquad r \ge Rs^4. \tag{2}$$

For verification use upper bounds A, X, B on the first three left sides. Split v into 96 equal intervals, maximizing each right side separately on that interval.

2. Filling the auxiliary rows using their remaining edges

Write $f = 7/30,\ \rho = f/s,\ \rho_0 = f/K,\ \gamma = A/f,\ g = X/f,\ \kappa = B/f$. Let G consist of exterior vertices with at most ρr target neighbors in each retained row, and let $\Delta = hr - |G|$. We prove

$$\Delta > r/8. \tag{3}$$

The same argument gives a separate restriction theorem. If $1 \le h \le r,\ r \ge 2^{20}$ and $h^2/r \le 100/3199$, no set of $hr+1$ vertices can have target independence at most h and independence at most r in every other color in a counterexample.

Inter-clique edges are undiscounted, so $4 \cdot 61^2 A/K < 1$ supplies independent representatives on arbitrary ordinary lists of size at least $r/61$, by the counting lemma in Appendix E. In particular independent $h+1$ vertices of G would extend. For the later temporary ordinary restriction remove the vertices whose foreign degree from unrecorded edges exceeds $r/44$. At most $44y_i'r$ points are removed from C_i. Each surviving point has at most ωr incident recorded edges, since each recorded source offers it one edge. Thus the temporary rows have size at least $255r/256$ and degree from each survivor outside its original clique at most $r/32$. The two numerical inequalities are $(g + 44\gamma)\rho_0 < 1/256$ and $1/44 + g\rho_0 + 1/256 + \omega < 1/32$.

For (3), suppose $\Delta \le cr$, $c = 1/8$. Take $hr+1$ points of G if available and otherwise take G. The target independence cap there is h. In the restriction theorem take the given $hr+1$ points instead. Apply the stability calculation of base §3 on these auxiliary points. Normalize edge counts by r^2 and lengths by r. Write $u_i = (r - |R_i^a|)_+/r$ for raw positive deficits of the auxiliary parts and x for their total interpart edge count in these units. With $I = 2^{-20}$ we have

$$x \le \xi, \qquad \sum u_i \le D, \qquad 2x + \sum u_i^2 \le T.$$

For (3) use $\xi = c^2 + I,\ D = c + \xi/[2(1-I)],\ T = (2 + 1/799)c^2 + 3I$. These are Appendix E's estimates using $h \ge 799,\ h^2/r < I$. The sum bound follows by charging the overfull parts to the other colors and adding the net deficit. For the restriction theorem use $\xi = 100/3199,\ D = \xi/[2(1-I)],\ T = 2\xi + I$. In fact the target excess over the balanced h-part floor on $hr+1$ is at most $r\binom{h}{2} - (r-h)$, by the other $r-1$ Turán floors. The balanced sum of squared deviations from r is one, and the net deficit is negative. These facts give the displayed bounds also when $h = 1$.

Prune auxiliary vertices of current interpart degree above $\tau_a r$. If the number pruned is zr, then $0 \le z \le x/\tau_a$ and the remaining interpart edge count is at most $x - \tau_a z$. The final normalized positive deficits have sum at most $D' = D + \xi/\tau_a$, and each is at most

$$\delta = D + (T - D^2)/(2\tau_a).$$

Indeed $u_i + x/\tau_a \le u_i + (T - u_i^2)/(2\tau_a)$, increasing up to $D < \tau_a$. For (3) take $(\tau_a, \theta_a, k) = (206/1000, 411/1000, 72/100)$. For the restriction take $(216/1000, 406/1000, 77/100)$. The surviving auxiliary parts are cliques. A nonedge in one would leave lists of fraction $1 - \delta - 2\tau_a > 0$ in the others, whose square exceeds ξ. Uniform choices therefore extend the pair to $h+1$ independent vertices.

Call an attachment large above $\theta_a r$ in an auxiliary row or above $\theta_o r$, $\theta_o = 369/1000$, in an ordinary row. We need extensions with one exterior point and at most **two** compatible prescribed representatives in

distinct rows, or with two independent exterior points and no representatives. The rows to be filled have only small attachments from these exterior points. Some rows may be omitted.

Fill auxiliary rows first. Fix two unfilled auxiliary rows with raw deficits u_1, u_2. For two exterior points put $w = 1-2\theta_a$. The initial lower bounds on their list fractions are $w-u_1, w-u_2$. Their product exceeds x, since

$$2(w-u_1)(w-u_2) + u_1^2 + u_2^2 = w^2 + (w-u_1-u_2)^2 \geq w^2 + (w-D)^2 > T.$$

Here $w > D$. For one exterior point put $w = 1-\theta_a$. The prescribed representatives can delete in total at most $2\tau_a r$ points across the auxiliary rows, since each auxiliary representative has this total interpart degree bound with factor τ_a, and an ordinary one has the smaller $r/32$ bound. In either case write $b_1 \leq b_2$ for the sorted numbers $w-u_i$, and set $t = 0$ in the two-exterior case and $t = 2\tau_a$ in the one-exterior case. The final product is at least $b_2(b_1 - t - z)$. This bound follows by charging both losses against the larger factor. At $z = 0$ it exceeds x, using $(1-\theta_a-D-2\tau_a)(1-\theta_a-D) > \xi$ in the second case. At $z = x/\tau_a$ it is positive, since $w - t > \delta$ and $u_i + x/\tau_a \leq \delta$. Linearity now gives a product greater than $x - \tau_a z$ at the actual pruning count. Each individual list is nonempty by the same endpoint bound. Consequently the expected number of target edges among independent uniform choices from the lists is less than one. This uses the total *remaining* edges just once. For ordinary rows, all the auxiliary choices cost at most $h\rho r \leq fr$, because they lie in G. Both $255/256 - 2\theta_o - f$ and $255/256 - \theta_o - f - 2/32$ exceed $1/61$. In the restriction theorem and in the $hr+1$ case only auxiliary rows are needed.

As in base §3, every exterior point must have a large row. Points with a unique large row join it completely, and two such points for the same row are adjacent. Otherwise the preceding extension produces one too many independent vertices. Adjoin these points to their rows; the enlarged rows remain cliques. For P remaining exterior vertices the enlarged deficits add to $P-1$, since the order is exactly one more than the number of rows times r. Each remaining point has at least two large rows before enlargement.

3. Edge counts at the destination row

Set $q_0 = 1/10000$. In an enlarged row with deficit dr, price a large arrival at $(kd+q_0)r$. Write $a = kd+q_0$ and $\theta = \theta_a$ or θ_o according to the row. If the row has $n > 0$ large arrivals, then $(n+1)/(2r) - d \leq a$. To see this let $t = 2(d+a) < 1$. With at least $\lfloor tr \rfloor$ arrivals, the one-clique capacity (base (4)) on that many actual vertices requires $\theta u < (1-u)d + u^2/2 + u/(2r)$, $u = \lfloor tr \rfloor / r$. At t the difference before rounding is

$$[2(\theta-a)(1+k) - 1]d + 2(\theta-a)q_0.$$

The bracket is positive and the second term exceeds $(\theta + d + 1/2)/r$, even at the maximum allowed deficit. This rules out the stated count and proves the price bound.

For a point with precisely two large rows let d_1, d_2 be their deficits **before** enlargement, in units of r. Write D_2 for an upper bound on $d_1 + d_2$. Use the individual caps δ for an auxiliary and $o = 1/256$ for an ordinary, and cap the sum over auxiliary rows by D'. If the two degrees together were at most $(1 + kD_2 + 2q_0)r$, the product of the two nonneighbor-list sizes would exceed their entire target cut, bounded by $r + d_1 d_2 r^2$. Explicitly, if d, e are the individual upper bounds, the two endpoint tests, with either row placed first, are

$$(1-d-\theta)(\theta - e - kD_2 - 2q_0) > \min(de, D_2^2/4) + I.$$

Each factor is positive. The minimum at fixed total degree is at an endpoint by concavity. A compatible pair would extend with the point. For three or more large rows, any three already contribute more than r net, without prescribing representatives. For j ordinary rows among the three the numerical comparison is

$$j\theta_o + (3-j)\theta_a - k\{jo + \min(D', (3-j)\delta)\} - 3q_0 > 1.$$

Each further arrival has positive net charge. The checker tests every ordered row type in both applications. The prices used in these tests dominate the prices after enlargement. Summing the one-clique capacities

gives $rP \le (r-1)(P-1)$, impossible. When $P = 0$ the nonnegative deficits already contradict their sum. This proves (3) and the restriction theorem.

4. The edge bound for uniquely assigned senders

Use the marks, retries and circular arcs of Appendix D, with their original target edges. Its symbols $N_i, n_i, E_i, F_i, Q, T_1, H_E$ denote respectively the number of uniquely strong senders at a row, its expected heavy-load count, its expected exterior cut, the incidences from foreign representatives, and the three quantities in its repair bound $\mathcal{R} = (4/3)Q + jT_1 + H_E$. Here $j = (1 + 80\rho_0)/(1 - g\rho_0)$. Write $\sigma = \sum d_i^2$, $w = \sigma/r$, $l = (\Delta - m)/(\rho r)$, where m is the expected number of marks. Thus $w \le \kappa\rho r$. Retained vertices have degree at most $r/4$ into every other retained clique, as required by the arcs.

In counting n_i, allow one sender for each recorded source at destination i. There are at most ωr there and pr in total. The represented points lie in true cliques, so all their inter-clique edges are charged in y'_i. Thus $F_i \le (2jg\rho_0)y'_i r^2 \le y'_i r^2/512$. The heavy-load bound in the 218 proof becomes

$$\frac{n_i}{r} \le x_i + (b_i)_+/r + \omega + \frac{513y'_i}{512\rho}, \qquad \sum_i n_i \le M_0 sr, \qquad M_0 = \frac{513Y_{\mathrm{hi}}}{512f} + \frac{7}{K},$$

where Y_{hi} is the band's upper endpoint. The first inequality counts all own-part senders separately. For each other sender outside the recorded group its heavy indicator is at most its load divided by ρr. In summing, use $18/5 + 1/2 + p_* < 7$. The repair inequality itself still uses the full actual E_i.

Here are the constants needed to replay the signed cut. Put $\eta = 1/(Rf^2)$, $z = 513\gamma/512 + 1/256 + g\rho_0 + \omega$, and

$$a = \min(7/2, (249/250)/z), \quad Y = (a - 4/3 - a^2\kappa/2)/2,$$
$$k_1 = (4/3)/Y, \qquad b_1 = j(1 + 1/k_1), \qquad V = aM_0 + 9b_1/K.$$

For any upper bound W with $\sum_i N_i d_i/r \le W\rho r$, Appendix D's tested-edge inequality gives $\mathcal{R} \le P(W, l)\rho^2 r^2$, where

$$P(W,l) = k_1(b_1W + l)^2\left(\tfrac{1}{2} + \frac{a^2\kappa}{4Y}\right) + k_1\eta\left(\frac{(1+V)^2 + V/K}{2} + \frac{1}{2K}\right).$$

Indeed $B_\pi + (\Delta - m) \le a\sqrt{\kappa Q} + (b_1W + l)\rho r$ for the test $\pi_i = (an_i + b_1N_i)/r$. Here $az + 2b_1g\rho_0 < 1$. The same-edge term pays for representative conflicts exactly as in (5)–(6) of that supplement. This estimate itself uses no assumption that the subsequent one-row bound is less than one.

The first pass uses $W = 2\kappa$, since $N_i \le 2d_i$. Set $Z_1 = 8P(2\kappa, 0) + 2\eta$. Let A_0 denote expected eligibility and $L_0 = \sum d_i + \Delta + 1 - A_0$ expected failures before thinning. When $A_0 > \Delta$, the base oversampling bound gives $\sum d_i - L_0 \le Z_1\rho^2 r$, since $l = 0$ and $Z_1\rho_0^2 < \min(c/128, 1/16)$. When $A_0 \le \Delta$, $L_0 \ge \sum d_i + 1$.

For a unique sender let q_y be its neighbor fraction in the assigned row. At zero moderate assignments, Appendix E's row calculation keeps an extra $37\rho r\sum_y(1 - q_y)$ on the left of its capacity bound. With $\lambda = 501/1000$, that left side, including rL_0, is at most $r\sum d_i + \lambda\sigma$. Also $\sum_{y \text{ unique at } i} q_y \le d_i/(1 - g\rho_0)$ and $d_i \le g\rho r$. Consequently

$$\frac{1}{r}\sum_i N_i d_i \le \frac{w}{1 - g\rho_0} + \frac{g}{37}\{\lambda w + \sum d_i - L_0\} \le W_1\rho r, \qquad W_1 = \left(\frac{1}{1 - g\rho_0} + \frac{\lambda g}{37}\right)\kappa + \frac{gZ_1\rho_0}{37}.$$

This uses the actual missing target edges that make retries succeed. A large number of successful retries would contradict the first oversampling bound.

Repeat the signed cut with W_1. Put $Z_2 = 8P(W_1, 0) + 2\eta$ and $\epsilon(l) = P(W_1, l) + gl + l^2/2 + \eta(1/2 + 1/(2K))$. Discarding the extra retry charge in the last capacity inequality gives $0 \le l \le \lambda\kappa$. By the one-row and

oversampling comparisons of Appendix D the required inequality is

$$(1-\epsilon(l))(c-\lambda\kappa\rho_0)+l-\lambda\kappa-Z_2\rho_0-\frac{1}{fRK^2}\le 0.$$

The checker verifies positive Y, both oversampling caps, the list and degree bounds, and $\epsilon<1$. The displayed left side is concave in l, so its minimum occurs at an endpoint. All $6\cdot 96\cdot 2=1152$ exact endpoint values exceed $1/2000$. This contradiction proves Theorem 1. Rational certificates and small-graph enumeration

Appendix D. Sampling the edges between rows

1. Choosing rows with less total deficit

Assume a counterexample and choose a sparsest color J. Each color has independence number at most r. Use the stability partition $R_1, \dots, R_r$ of base §1. Throughout, edges and neighborhoods without a color specified refer to J. Let $b_i = |R_i| - r$, let μ_i count internal pairs missing J, and let X count edges between parts. The base gives $\sum b_i = 1$, $\sum (b_i)_+ \le r/2$, $\sum \mu_i \le \binom{r}{2}$, $X \le r(r-1)$.

Put $U_i = \mu_i - (r-1)b_i$. Each of the other $r-1$ colors needs at least b_i edges inside R_i, since deleting one endpoint per edge leaves an independent set. Thus $U_i \ge 0$ and $\sum U_i \le r^2/2$. For any z exterior points Z_i, the same count in $R_i \cup Z_i$ gives

$$e_J(Z_i, R_i) \le U_i + (1 + b_i)z + \binom{z}{2}. \tag{1}$$

Indeed all edges of the other colors must fit among the μ_i internal pairs, the nontarget cross pairs and the pairs of Z_i.

A part qualifies as a destination if $b_i \le r/256$ and $U_i \le r^2/1200$. At most $128 + 600$ parts fail. In a qualifying part, $z \le \lfloor r/256 \rfloor$ exterior points each sending more than $r/4$ edges into it would cost $U_i > (61/250)zr$, by (1). At $z = \lfloor r/256 \rfloor$ this exceeds the qualifying budget for $r \ge 2^{20}$. Hence the total number of such exterior points is at most $U_i/((61/250)r)$.

First delete a vertex while it has current degree above $r/4$ into a single qualifying foreign part. Then delete a vertex while its total current interpart degree exceeds τr, where $\tau = 47/100$. This changes the thresholds of the earlier two-stage pruning argument. Let pr, tr be the two removal counts. Each removal destroys fresh edges, and every first-stage removal spent an original star counted by (1). Earlier pruning argument

Consequently

$$0 \le p \le 125/61, \quad t \ge 0, \quad p/4 + \tau t \le 1, \qquad Z = 1 - p/4 - \tau t.$$

There are at most Zr^2 cross edges left. Trim oversized parts to r. The total deficit before selecting pairs is $D_0 \le (1/2 + p + t)r$, by the bound on $\sum (b_i)_+$.

Take a maximal family of q pairs in distinct parts, with all $2q$ endpoints jointly independent. In every unchosen part remove the neighbors of these endpoints. At most $2\tau qr$ vertices are lost. Each unchosen survivor list is a clique by maximality. Write m_i for its size. In both experiments the draws in distinct lists are independent. First select one point uniformly from r positions in each unchosen list, putting empty positions after its m_i points. Keep the real selections and delete one endpoint per conflicting edge. The expected number of conflicts is at most Z. The selected survivors and the $2q$ fixed endpoints are independent, so

$$q \le \sum_{\text{unchosen}} (1 - m_i/r) + Z \le D_0/r + 2\tau q + Z.$$

This holds for every maximal family, including one chosen without optimizing its size.

Set $\ell_0 = 19/20$ and omit the unchosen lists of size below $\ell_0 r$. In a second experiment select a uniform point from each long list. In a short list select with probability $m_i/(\ell_0 r)$, uniformly when nonempty. Every individual point now has selection probability at most $1/(\ell_0 r)$. The same conflict deletion proves

$$q \le \sum_{\text{short}} (1 - m_i/(\ell_0 r)) + Z/\ell_0^2.$$

Each omitted short list accounts for a deficit $r - m_i \ge 2\tau r(1 - m_i/(\ell_0 r))$. Subtract these deficits from $D_0 + 2\tau qr$. Omit the chosen parts and every unqualified destination as well. For the remaining cliques

C_i, write $d_i = r - |C_i|,\ x_i = d_i/r,\ y_i = e_J(C_i, V \setminus R_i)/r^2$. We have

$$\sum x_i \le L_x = \tfrac{1}{2} + p + t + 2\tau Z/\ell_0^2, \qquad \sum y_i \le L_y = 2 - p/4 - \tau t. \tag{2}$$

The second bound subtracts the spent incidences at deleted endpoints from the original $2X$. First-stage pruning bounds the degree of each survivor into each other retained clique by $r/4$. This per-destination bound suffices for the circular arcs of base §4: a sender has at most one representative, and its load is tested in one destination at a time.

The domain of p, t is a trapezoid. Writing $p_* = 125/61$, its corners are $(0,0), (p_*, 0), (0, 1/\tau), (p_*, (1 - p_*/4)/\tau)$. At all four, $L_x < 18/5$, $(1/2+p+t+Z)/(1-2\tau) < 60$, and $(1/2+p+t+2\tau\cdot 60)/(1-\ell_0) < 1200$. These affine bounds hold throughout, giving fewer than 60 chosen slots and 1,200 short ones. Thus the initial number e omitted is at most 1988.

For $r \ge 2^{36}$, even the first $n = 2^{33}$ entries of the proposed profile have total normalized deficit at least

$$\sum_{i=1}^{n} \frac{\lfloor 4r/(25i) \rfloor}{r} \ \ge\ \frac{4}{25} H_n - \frac{1}{8} \ >\ \frac{4}{25}(33 \log 2 + 49/100) - \frac{1}{8} \ >\ \frac{18}{5}.$$

Here $H_n \ge \log(n+1) + (1 - 1/(n+1))/2$, by the trapezoid bound for the integral of $1/x$. This proves Theorem 2.

2. Choosing the width

Put $K = 2500,\ R = 2^{19},\ f = 7/30,\ \alpha = 1/6,\ \beta = 11/60,\ S = 2^{199/4}$. As in Appendix F, give a row weight $a_i = \max(y_i, \alpha x_i/(2\beta))$. Set

$$s(z) = \max\{K, e + 1 + \#\{a_i > z\}\}, \qquad C(z) = \sum_{a_i \le z} x_i^2/z.$$

Integrating $s(z)z/\alpha + \alpha C(z)/(4\beta^2)$ against dz/z from α/S to α/K costs at most $1 + L_y/\alpha + L_x/\beta$. The constant K costs at most one; each row costs at most $y_i/\alpha + x_i/\beta$ by integration on either side of its weight. At the four corners of §1 this upper bound is at most $12887461/484462 < \log(S/K)$. For the strict comparison use $(155/4)\log 2 - \log(625/512) > (155/4)(6931/10000) - 113/512$.

Choose a level at which the integrand is below one. Omit the rows above it and enough additional slots to omit exactly $h = s - 1$, where $s = s(z)$. With $v = sz/\alpha \in (0,1)$, the simultaneous bounds on the retained rows are

$$s \max y_i \le \alpha v, \qquad s \max x_i \le 2\beta v, \qquad s \sum x_i^2 \le 4\beta^2 v(1-v). \tag{3}$$

The last bound follows by multiplying $C(z) < 4\beta^2(1-v)/\alpha$ by sz. In particular $K \le s < S$ and $r \ge Rs^4$ when $r \ge 2^{218}$. Keeping v in (3) means a large pointwise cut bound and the largest possible squared deficit cannot both occur at the chosen level.

3. The exterior shortage

In §§3–4 let $s \ge s_0 \ge 800,\ \rho = f/s,\ \rho_0 = f/s_0$, and suppose $r \ge R_0 s^4$. Write G for the exterior points sending at most ρr edges to every ordinary clique. Put $\Delta = hr - |G|$. In the two applications below, the ordinary lists, temporary outside-degree bounds and arc extensions of Appendix E hold. Section 5 supplies their exact numerical checks. We strengthen its shortage to

$$\Delta > cr, \qquad c = 89/1000. \tag{4}$$

Suppose $\Delta \le cr$. The auxiliary construction in §2 of Appendix E applies to a subset of G of size $hr + 1$ if available, and to G otherwise. In either case $h \ge 799,\ h^2/r < 2^{-20},\ r \ge 2^{20}$. Put $\xi = c^2 + 2^{-20}$. A

raw auxiliary deficit divided by r is bounded above by $d_* = c + \xi/(2(1-2^{-20}))$, since the other colors pay for the oversized auxiliary parts. For a raw positive deficit d in units of r, twice the auxiliary cross edge count divided by r^2 is at most $M_* - d^2$, where $M_* = (2+1/799)c^2 + 3\cdot 2^{-20}$. Prune at interpart degree $\tau_a r$, with $\tau_a = 27/200$. The final normalized deficit is at most $d_* + (M_* - d_*^2)/(2\tau_a) < \delta = 3/25$, since $d_* < \tau_a$. The auxiliary rows are cliques and their $r/10$ lists have independent representatives by $1-\delta-2\tau_a > 1/2,\ 100\xi < 1$.

Use the ordinary temporary restriction of Appendix E, of size at least $255r/256$ and outside-clique degree at most $r/32$. Set $\theta = 93/250$ for a large attachment. In the extension from base §3 we need at most **three** prescribed representatives, or two independent exterior points. Indeed the auxiliary fractions $1-\delta-\theta-3\tau_a$ and $1-\delta-2\theta$ exceed $1/10$. The ordinary fractions $255/256-\theta-f-3/32$ and $255/256-2\theta-f$ exceed $1/61$. As in the base proof, every exterior point has a large attachment. Those having a unique one form a clique with their respective row and may be adjoined there. The remaining P_0 exterior points each have at least two large attachments before enlargement, and the enlarged row deficits sum to $P_0 - 1$.

For an enlarged row with normalized deficit $d \in [0,\delta]$, let z count assigned large arrivals (one per large attachment). Retain the dr contribution per arrival in the other-color capacity inequality (base (4)). We claim $(z+1)/(2r) - d \le a_0 = 53/500$ whenever $z > 0$. Otherwise there are at least $\lfloor tr \rfloor$ arrivals with $t = 2(d+a_0)$. For $u = \lfloor tr \rfloor / r$, their degrees exceed θr and capacity would give $\theta u < (1-u)d + u^2/2 + u/(2r)$. But

$$\theta t - (1-t)d - t^2/2 = 2(\theta - a_0)(d + a_0) - d \ \ge\ 2(\theta - a_0)(\delta + a_0) - \delta.$$

The right side exceeds $(\theta + 2\delta + a_0)2^{-20}$, covering the rounding and linear term. Thus $z < \lfloor tr \rfloor$, which gives the claim. The net cost per arrival on the capacity side, after using its own deficit, is at most $a_0 r$.

For a point with two large attachments, if their normalized degree sum before enlargement is at most $1 + 2a_0$, its two nonneighbor lists have product at least $r^2(1-\delta-\theta)(\theta-\delta-2a_0) > r + \delta^2 r^2$. This is the concave-product endpoint bound of base §3; it exceeds the actual two-clique cut, so a compatible pair would extend. With three large attachments, if all three lists have size at least $21r/100$, the expected number of conflicts is below one by $3(1/r + \delta^2) < (21/100)^2$. Otherwise the normalized net charge exceeds $1-\delta-21/100+2\theta-3a_0 > 1$. Four or more pay at least $4(\theta - a_0) > 1$. Summing the row capacities would now give $rP_0 \le (r-1)(P_0 - 1)$. This proves (4), including the case $P_0 = 0$ which already contradicts the deficit sum.

4. Conflict bounds from the tested edges

We give the changed part of the sampling proof. Write $s \max y_i \le A,\ s \max x_i \le X_0,\ s\sum x_i^2 \le B,\ \sum y_i \le Y_0$. Put $\gamma = A/f,\ g = X_0/f,\ \kappa = B/f,\ w = \sum d_i^2/r$. Then $w \le \kappa\rho r$. In both applications $\sum d_i \le 9r/2,\ \sum (b_i)_+ \le r/2$. All definitions of marks, loads and arcs are as in §§3–4 of Appendix E. We recall those needed here.

The strong senders have degree greater than $r/2$ in a row. Those with a unique strong row get one uniform draw there and may retry with probability 80ρ after drawing a neighbor. Let N_i count unique strong senders in row i. Write m for the expected number of marks, $u = \Delta - m$, and $l = u/(\rho r)$. The two cases of Appendix E use

$$(U_*, \lambda_*) = (41/40, 501/1000) \text{ or } (2, 0).$$

They give $\sum N_i d_i \le U_* \sum d_i^2,\ 0 \le l \le \lambda_* \kappa,\ m \ge (c - \lambda_* \kappa \rho_0) r,\ m \le sr$. The second case uses the retry surplus when $\sum N_i d_i > (41/40)\sum d_i^2$. The checks on g, ρ_0 in §5 ensure that surplus pays the strong-row cost.

For the marked exterior set together with G, let E_i be its expected cut into C_i, F_i the expected incidences from representatives in other rows, and n_i the expected count of loads above ρr. Put

$$Q = \sum n_i (E_i + F_i)/r, \quad Q_F = \sum n_i F_i / r, \quad T = \sum N_i E_i / r.$$

A specified vertex has representative probability at most j/r per unique sender in its row, with $j = (1+80\rho_0)/(1-g\rho_0)$. Put $M = (513/512)Y_0/f + 5/s_0$. As before,

$$n_i/r \le x_i + (b_i)_+/r + 513y_i/(512\rho), \qquad \sum n_i \le Msr, \qquad \sum n_i d_i/r \le \sqrt{\kappa Q}.$$

These follow respectively from $2jg\rho_0 \le 1/512$, summation, and Cauchy–Schwarz.

Let $e_{ik} = e_J(C_i, C_k)$ for $i \ne k$. Define $H_E = j^2 \sum_{i<k} N_i N_k e_{ik}/r^2$. This bounds distinct-row representative conflicts by independence. The within-row conflict count and circular arcs from Appendix F give an upper bound

$$\mathcal{R} = (4/3)Q + jT + H_E$$

for the expected repair cost. Counting each possible representative endpoint on the very same cross edges also gives

$$Q_F \le \frac{j}{r^2} \sum_{i<k} (n_i N_k + n_k N_i) e_{ik}. \tag{5}$$

Test rows independently with probabilities $\pi_i = (an_i + bN_i)/r$, independently of the sample. The signed cut identity in base §5 has the actual target edges **between** tested rows on its right side. Keeping their expectation there gives

$$\sum_i \pi_i E_i + \sum_{i<k} \pi_i \pi_k e_{ik} \le \mathcal{R} + r\,\mathbb{E}\binom{H}{2} + (B_\pi + u)^2/2 + m/2,$$

where $H = h + \#\{\text{tested rows}\}$ and $B_\pi = \sum \pi_i d_i$. The first sum equals $a(Q - Q_F) + bT$. For $b \ge j$, expansion on each edge and (5) give

$$\sum_{i<k} \pi_i \pi_k e_{ik} \ge aQ_F + (b/j)^2 H_E.$$

Thus both the representative loads and representative-pair conflicts can be paid on their actual edges. Combined, these estimates yield

$$(a - 4/3)Q + (b-j)T + ((b/j)^2 - 1)H_E \le r\,\mathbb{E}\binom{H}{2} + (B_\pi + u)^2/2 + m/2. \tag{6}$$

For the finite numerical certificate put $\eta = 1/(R_0 f^2)$ and choose

$$\begin{aligned} a &= \min\left(\tfrac{7}{2}, \frac{249/250}{513\gamma/512 + 1/256 + g\rho_0}\right), & D_a &= a - 4/3 - a^2\kappa/2, \\ Y &= 9D_a/20, \qquad k = (4/3)/(D_a - Y), & b &= j(1 + 1/k), \\ V &= aM + 9b/s_0, & B_0 &= ((1+V)^2 + V/s_0)/2. \end{aligned}$$

Section 5 checks $D_a > 0$ and $a(513\gamma/512 + 1/256 + g\rho_0) + 2bg\rho_0 < 1$; hence the testing probabilities are valid. We have $\mathbb{E}\binom{H}{2} \le B_0 s^2$ and $B_\pi + u \le a\sqrt{\kappa Q} + (bU_*\kappa + l)\rho r$. Use Young's inequality on the mixed term with coefficient Y on Q, then multiply (6) by k. The resulting coefficients dominate $4/3, j, 1$ on Q, T, H_E, respectively. Since $s^2 r \le \eta \rho^2 r^2$, this gives

$$\begin{aligned} \mathcal{R} &\le P(l)\rho^2 r^2, \\ P(l) &= k(bU_*\kappa + l)^2 \left(\tfrac{1}{2} + \frac{a^2\kappa}{4Y}\right) + k\left(B_0 + \frac{1}{2s_0}\right)\eta. \end{aligned}$$

For one tested row, the signed cut adds at most $ud_i + u^2/2 + m/2 + r\binom{s}{2}$ beyond repairs. Therefore its

expected moderate assignment count is at most $\epsilon(l)\rho r$, where

$$\epsilon(l) = P(l) + gl + l^2/2 + \eta(1/2 + 1/(2s_0)).$$

When oversampling is necessary $l = 0$. At up to twice the original thinning probability the same three conflict terms are bounded by $4\mathcal{R}$, since $Q' \le 4Q$, $T' \le 2T$, and H_E bounds representative-pair conflicts at either probability. The oversampling floor of base §5 and the final row inequality of Appendix E now require

$$(1 - \epsilon(l))(c - \lambda_*\kappa\rho_0) + l - \lambda_*\kappa - (8P(0) + 2\eta)\rho_0 - \frac{1}{fR_0 s_0^2} \le 0. \tag{7}$$

Specifically $Z_0 = 8P(0)+2\eta$ bounds the oversampling excess divided by $\rho^2 r$, since $Z_0\rho_0^2 < \min(c/128, 1/16)$. The lower bound on failed senders retains u and the moderate quadratic sum is at most $\epsilon(l)\rho r m$, as in (6) of Appendix E. The left side of (7) is concave in l; it suffices to check $0, \lambda_*\kappa$ in the first branch and 0 in the second.

5. Exact certificates and the earlier mixed profile

For Theorem 1 use $s_0 = 2500$, $R_0 = 2^{19}$, $Y_0 = 2$. For each $i = 0, \dots, 63$, bound the level parameter v from (3) on $[i/64, (i+1)/64] = [v_-, v_+]$. Set

$$u_* = \max(v_-, \min(1/2, v_+)), \qquad A = \alpha v_+, \quad X_0 = 2\beta v_+, \quad B = 4\beta^2 u_*(1 - u_*).$$

These upper bounds apply throughout the interval, including when κ is small. The checker verifies the ordinary transversal test $4 \cdot 61^2\gamma\rho_0 < 1$, temporary row sizes and outside degrees, the circular-arc list fraction, both retry bounds, validity of the row tests, $\epsilon < 1$, the oversampling cap, and (7). Every one of the 192 rational endpoint values on the left of (7) exceeds 1/1000. This contradiction proves Theorem 1. Exact arithmetic

Appendix E. Counting independent transversals

1. Parameters and a count of independent transversals

Assume every color graph has independence number at most r. Use the notation of Appendix F. The ordinary cliques C_i lie in original parts R_i. Write $b_i = |R_i| - r$, $d_i = r - |C_i|$, $x_i = d_i/r$, $y_i = e_J(C_i, V \setminus R_i)/r^2$. Before the width selection, at most 1,777 parts have been omitted. All remaining $b_i \le r/256$. Write

$$K = 1800, \quad R = 2^{19}, \quad f = 7/30, \quad \alpha = 29/250, \quad \beta = 19/125, \quad S = 2^{135/2}.$$

Use the level selection in §1 of Appendix F with these α, β, K, S. Its integral bound is $1 + 1/\alpha + 9/(2\beta) < \log(S/K)$. Indeed $1/\beta > 1/(4\alpha)$ and $\log(S/K) = (113/2)\log 2 + \log(256/225) > (113/2)(6931/10000) + 31/256$. We obtain $K \le s < S$, $h = s - 1$, $\rho = f/s$, $\rho_0 = f/K$. For the $r - h$ remaining rows, put $D = \sum d_i$, $\sigma = \sum d_i^2$, $w = \sigma/r$. The guarantees are

$$\gamma = \alpha/f, \quad g = 2\beta/f, \quad \kappa = \beta^2/f, \qquad y_i \le \gamma\rho, \quad d_i \le g\rho r, \quad w \le \kappa\rho r, \quad D \le 9r/2, \quad r \ge Rs^4. \tag{1}$$

The last bound uses $19 + 4(135/2) = 289$.

For lists of size at least L, suppose each list meets at most $L^2/4$ forbidden pairs to the other lists, counted in total. Any subfamily has independent representatives. In fact its number $Z(T)$ of independent transversals is at least $(L/2)Z(T \setminus \{i\})$ for each $i \in T$. Induct on $|T|$, starting with $Z(\emptyset) = 1$. Of at least $LZ(T \setminus \{i\})$ extensions, a forbidden pair to row j rules out at most $Z(T \setminus \{i, j\}) \le (2/L)Z(T \setminus \{i\})$. Summation proves the claim, also for a real lower bound L. We use $L = r/61$, since $4 \cdot 61^2\alpha/K < 1$.

2. A stronger shortage of exterior points

Let G be the exterior points sending at most ρr edges of J to each ordinary row and put $\Delta = hr - |G|$. We claim

$$\Delta > cr, \qquad c = 13/200. \tag{2}$$

We detail the changes to base §3. Its proof also gives (2) for the rows of Theorem 2 under the list and size conditions checked in §5. In both applications $h \ge 1799$, $r \ge 2^{20}$, $h^2/r < 2^{-20}$. Independent $h + 1$ points of G leave lists of size $r/61$ and would extend. If $\Delta \le cr$, take $X_0 \subseteq G$ of size $hr + 1$ when possible and otherwise take $X_0 = G$. Write $|X_0| = hr + b$. Either $b = 1$ or $b = -\Delta$ with $0 \le \Delta \le cr$.

For the h auxiliary parts on X_0, let η_0 be its target edge excess over the Turán floor and $\mu_0 \le \eta_0$ the number of internal pairs missing J. As in base §3, $2\eta_0/r^2 \le \xi = c^2 + 2^{-20}$. Each of the other $r - 1$ colors needs at least $(|B| - r)_+$ edges in each auxiliary part B, by deleting an endpoint per edge. Consequently $\sum_B (|B| - r)_+ \le \eta_0/(r - 1)$. Adding the net deficit $-b$, every raw positive deficit divided by r is at most

$$d_* = c + \frac{\xi}{2(1 - 2^{-20})} < 0.06712.$$

If X_a counts interpart target edges and $d = (r - |B|)_+/r$, the stability calculation gives

$$2X_a/r^2 + d^2 \le M_* = (2 + 1/1799)c^2 + 3 \cdot 2^{-20}, \qquad X_a/r^2 \le \xi.$$

Indeed the sum of squared deviations of the actual parts plus $2X_a$ is at most $4\eta_0$ plus the balanced sum of squared deviations, at most $b^2/h + h/4$. Delete vertices of current interpart degree above τr, where $\tau = 27/250$. A part's final normalized deficit is at most $d + (M_* - d^2)/(2\tau)$. This expression is increasing on $[0, d_*]$, and its value at d_* is below $\delta = 43/500$.

Base §3 now gives auxiliary cliques with arbitrary lists of size $r/10$ admitting independent representatives. Its two tests are $1 - \delta - 2\tau > 1/2$ and $100\xi < 1$. In the temporary ordinary restriction, the minimum

surviving size is $255r/256$ and each survivor has degree at most $r/32$ outside its original clique. For (1) these follow from $1/40 + g\rho_0 + 1/256 < 1/32$ and $(g + 40\gamma)\rho_0 < 1/256$. For the prescribed representatives the auxiliary test is $1 - \delta - 93/250 - 4\tau > 1/10$. The other list bounds and the attachment comparisons in base §3 keep their parameters $\theta = 93/250,\ t = 133/500$; the exact checker includes each. The resulting attachment contradiction proves (2). For the circular-arc argument with (1) we also have $1-g\rho_0-3/4-f > 1/61$.

3. Using the labels of edges into a row

The other colors constrain every set of z exterior vertices of target degrees j_y into a clique of deficit d. Appendix F, (4)–(5), gives

$$\sum_y (j_y + d) \le (r-1)d + z(z+1)/2, \qquad b^2/2 + (b-d)v_0 - J_d \le \lambda d^2 + v_0^2/2, \tag{3}$$

where $\lambda = 501/1000,\ b$ is the number of strong assignments, $q = j_y/(r-d)$, $J_d = \sum_{\text{strong}} 2d(1-q)$, and $v_0 \ge 0$. Here a vertex with degree greater than $r/2$ in more than one row is assigned to two of them. A vertex with exactly one such row is assigned there. The first inequality gives $b \le 2d,\ \sum_{\text{strong}} j_y \le rd$; the second uses $2g\rho_0/(1 - g\rho_0) < 1/1000$.

For each uniquely assigned sender, if a first uniform draw in its row hits a target neighbor, retry with probability 80ρ using an independent uniform draw. Accept the first nonneighbor. The success probability is $v = (1-q)(1+80\rho q)$. After paying $r(1-v) + \rho r v + 2d(1-q)$ from $j_y + d$, the amount left is

$$(1-q)\{r\rho[80q(1-\rho) - 1] - d\} \ \ge\ 37\rho r(1-q).$$

This uses $q > 1/2$ and $40(1-\rho_0) - 1 - g > 37$. Each assignment of a multiply strong sender still pays at least $r/2 + 2d(1-q)$. Lean proof of the retry calculation

Let N_i be the number of uniquely assigned senders in row i, and let M_i be the sum of their $1-q$. Put $U = \sum N_i d_i/\sigma$, taking $U = 0$ if $\sigma = 0$. Thus $0 \le U \le 2$. Since the sum of the q's of the unique assignments in row i is at most $d_i/(1 - g\rho_0)$, the total extra charge satisfies

$$37\rho r \sum_i M_i \ \ge\ \frac{37}{g} \sum_i d_i M_i \ \ge\ \frac{37}{g}(U - 1/(1 - g\rho_0))\sigma.$$

When $U > 41/40$ this pays the entire $\lambda\sigma$ in the sum of (3). Use the bounds $(U_*, \ell) = (41/40, \lambda)$ for $U \le 41/40$, and $(U_*, \ell) = (2, 0)$ otherwise. In both cases $\sum N_i d_i \le U_*\sigma$.

All successful uniques and all moderate senders are eligible. Let A be their expected count and $L_0 = D + \Delta + 1 - A$. Applying (3) at $v_0 = 0$ gives $rL_0 \le rD + \ell\sigma$. Mark eligible senders independently with probability $\min(1, \Delta/A)$, with all draws independent across senders. Here $A \ge \Delta + 1 - \ell w > 0$. For the expected number of marks m, put $u = \Delta - m = (\Delta - A)_+$ and $\vartheta = u/(\rho r)$. Then

$$0 \le \vartheta \le \ell\kappa, \qquad (c - \ell\kappa\rho_0)r \le m \le sr.$$

Let v_i be the expected number of marks from moderate senders assigned in advance to row i of degree above ρr. Applying (3) to the strong senders and the marked moderates gives, as in §5 of Appendix F,

$$rL_0 + \rho r m \le rD + \ell\sigma + \sum_i v_i^2 + m. \tag{4}$$

Indeed the extra charge just proved is available also at $v_0 = v_i$. The variance and linear cost of the moderate count sum to at most m.

4. Counting repairs and retaining the unsampled shortage

Write $W = G \cup \{\text{marked senders}\}$. Use E_i, F_i, n_i, Q, Q_F, T from §4 of Appendix F. Thus E_i counts expected target edges between C_i and W, F_i counts expected target incidences from other rows' representatives, and n_i counts expected loads exceeding ρr. In particular

$$Q = \sum n_i(E_i + F_i)/r, \quad Q_F = \sum n_i F_i/r, \quad T = \sum N_i E_i/r.$$

At any thinning probability the chance of a specific representative point for one sender is at most j/r, where $j = 1011/1000 > (1 + 80\rho_0)/(1 - g\rho_0)$. Put $M = 2(513/512)/f + 5/K$, $\chi = j^2 U_*^2/2$. The bounds on heavy loads and on the number P_* of added repair edges are

$$\begin{aligned} n_i/r &\le x_i + (b_i)_+/r + 513y_i/(512\rho), \quad \sum n_i \le Msr, \\ B_n = \sum n_i d_i/r &\le \sqrt{\kappa Q}, \qquad\qquad Q_F \le 10Msr + jU_* w\sqrt{\kappa Q}, \\ \mathbb{E}P_* &\le (4/3)Q + jT + \chi w^2 + 243r. \end{aligned} \tag{5}$$

For the first line $F_i \le y_i r^2/512$ follows from $2jg\rho_0 < 1/512$. For the second, $\rho r n_i \le E_i + F_i$ gives Cauchy–Schwarz, and $e_J(C_i, C_k) \le r + d_i d_k$ gives $F_i \le 10r + jU_* w d_i$, since $\sum N_i \le 9r$. The same two-clique bound on distinct-row representative pairs contributes at most $243r + \chi w^2$. Same-row and representative-to-exterior conflicts together contribute jT by the same-row count in Appendix F. The $4Q/3$ comes from intersecting arcs. Thus the right side of (5) bounds repairs at the base probability and the corresponding expression at each new thinning probability.

Test rows independently with probabilities $\pi_i = (an_i + bN_i)/r$, where $a = 199/100$, $b = 1217/1000$. These are at most one by (5) and $a(513\gamma/512 + 1/256 + g\rho_0) + 2bg\rho_0 < 1$. Put

$$V = aM + 9b/K, \quad B_0 = ((1+V)^2 + V/K)/2, \quad Y = 19/100, \quad k = 99/20.$$

In the expected signed cut inequality (base §5), the left side is $a(Q - Q_F) + bT$, the tested-row binomial costs at most $B_0 s^2 r$, and the quadratic cost is at most $(B_\pi + u)^2/2$ with a further $m/2$. Here $B_\pi = \sum \pi_i d_i \le a\sqrt{\kappa Q} + bU_* w$.

For $0 \le \vartheta \le \ell\kappa$ write $t = bU_* + \vartheta/\kappa$, $\eta = 1/(Rf^2)$, and define

$$\begin{aligned} C_w(t) &= k\left(\chi + t^2/2 + \frac{a^2(t + jU_*)^2\kappa}{4Y}\right) + \chi, \\ C_s &= k\,(B_0 + (10Ma + 1/2)/K + 243/K^2) + 243/K^2, \\ P(\vartheta) &= C_w(t)\kappa^2 + C_s\eta, \\ \epsilon(\vartheta) &= P(\vartheta) + g\vartheta + \vartheta^2/2 + \eta(1/2 + 1/(2K)). \end{aligned}$$

To verify these expressions when w is small, use $w \le \kappa\rho r$ **before** combining it with $u = \vartheta\rho r$. Thus $B_\pi + u \le a\sqrt{\kappa Q} + t\kappa\rho r$. Young's inequality bounds the resulting mixed term from the cut and aQ_F by $YQ + a^2(t + jU_*)^2\kappa(\kappa\rho r)^2/(4Y)$. The remaining coefficients of Q, T are $a - 4/3 - a^2\kappa/2 - Y$, $b - j$; after multiplication by k they exceed $4/3, j$. Using $s^2 r \le \eta\rho^2 r^2$ gives

$$(4/3)Q + jT + \chi w^2 + 243r \le P(\vartheta)\rho^2 r^2.$$

For a single tested row the cut costs $ud_i + u^2/2 + m/2 + r\binom{s}{2}$ beyond the repairs. Hence $E_i \le \epsilon(\vartheta)\rho^2 r^2$ and $v_i \le \epsilon(\vartheta)\rho r$.

Put $Z = 8P(0) + 2\eta$, using the branch's U_*. If $A > \Delta$, then $u = 0$. Oversample as in base §5 by $z' = \min(A - \Delta, \Delta/128, r/16)$. The doubled thinning bound $Q' \le 4Q$, $T' \le 2T$ and the floor $rz'/2 - rs^2$ give $z'/r \le Z\rho^2 < \min(c/128, 1/16)$, forcing $z' = A - \Delta$. If $A \le \Delta$, the identity $L_0 = D + 1 + u$ holds.

Combining the two cases retains the full linear term

$$L_0 \geq D + u - Z\rho^2 r.$$

Since $\sum v_i^2 \leq \epsilon(\vartheta)\rho r m$, (4) now requires

$$(1-\epsilon(\vartheta))(c-\ell\kappa\rho_0)+\vartheta-\ell\kappa-Z\rho_0-\frac{1}{fRK^2} \;\leq\; 0. \tag{6}$$

The function ϵ is increasing, convex and less than one throughout each interval by the endpoint bounds below. Therefore the left side of (6) is concave and takes its minimum at an endpoint. All entries below are strict rational comparisons. Checker for (1)–(6)

U_*	ϑ	Upper bound on ϵ	Lower bound on the left side of (6)
41/40	0	229/1000	3/10000
41/40	$\lambda\kappa$	395/1000	39/1000
2	0	789/1000	12/1000

Each lower bound contradicts (6). This proves Theorem 1.

Appendix F. Choosing ordinary rows and counting repairs

1. Selecting the ordinary rows

Assume a counterexample and fix a sparsest hue J. All degrees and cuts below count J-edges. Every color has independence number at most r. By §1 of Appendix G, on all r^2+1 vertices there are r original parts R_i, with $b_i = |R_i| - r$, $O = \sum (b_i)_+ \le r/2$ and at most $X \le r(r-1)$ edges between parts. In the pruning step $P \le 4X/r$ vertices are removed at current cross degree greater than $r/4$. Outside at most 1,649 omitted parts one obtains cliques $C_i \subset R_i$. Set $d_i = r - |C_i|$, $x_i = d_i/r$ and $y_i = e_J(C_i, V \setminus R_i)/r^2$. The extraction gives

$$\sum x_i \le P/r + 1/2, \qquad \sum y_i \le 2X/r^2 - P/(4r) \le 2 - P/(4r). \tag{1}$$

For the first bound, §1 pays for the neighbors of the independent pairs by omitting the short slots, leaving total deficit at most $P+O-1$. For the second, at least $rP/4$ of the original $2X$ cross incidences belong to removed vertices. Each retained vertex has at most $r/4$ neighbors in the other retained cliques together. Omit also the parts with $b_i > r/256$. There are at most 128 of these, so the total omitted count e is at most 1,777.

Fix the constants

$$K = 2160, \quad R = 2^{19}, \quad f = 7/30, \quad \alpha = 8/75, \quad \beta = 33/250,$$
$$\gamma = \alpha/f, \qquad g = 2\beta/f, \qquad \kappa = \beta^2/f.$$

Assign each retained row weight $a_i = \max\{y_i, \alpha x_i/(2\beta)\}$. For $z > 0$, put

$$s(z) = \max\{K, e+1+\#\{i : a_i > z\}\}, \quad A(z) = s(z)z, \qquad C(z) = \sum_{a_i \le z} x_i^2/z.$$

Integrate $A/\alpha + \alpha C/(4\beta^2)$ with measure dz/z from α/S to α/K, where $S = 2^{301/4}$. Since $e+1 \le K$, the constant K contributes at most one. A row contributes at most $a_i/\alpha + \alpha x_i^2/(4\beta^2 a_i)$, with zero contribution for $a_i = 0$. This follows by integrating the count below a_i and the squared term above a_i. If $a_i = \alpha x_i/(2\beta)$, the displayed cost is x_i/β. Otherwise the first term is y_i/α and the second is at most $x_i/(2\beta)$. By (1), $P/r \le 4$, and $1/\beta > 1/(4\alpha)$, the integral is at most

$$1 + \frac{2 - P/(4r)}{\alpha} + \frac{P/r + 1/2}{\beta} \le 1 + \frac{1}{\alpha} + \frac{9}{2\beta} < \log(S/K). \tag{2}$$

Indeed $K = 2^{11}(135/128)$, $\log 2 > 6931/10000$, and $\log(135/128) < 7/128$. Thus at some level the integrand is below one. There $A < \alpha$ and $AC < \beta^2$ by the arithmetic–geometric mean inequality, so $s = s(z) < S$.

Omit the rows above the level and enough additional rows to omit exactly $h = s-1$ parts in total. There are enough since $s < r$. Write $\rho = f/s$, $\rho_0 = f/K$, $D = \sum d_i$, $\sigma = \sum d_i^2$, $w = \sigma/r$ over the remaining ordinary rows. We have

$$y_i \le \gamma\rho, \qquad x_i \le g\rho, \qquad w \le \kappa\rho r, \qquad D \le 9r/2, \qquad r \ge Rs^4. \tag{3}$$

The last inequality follows from $19 + 4(301/4) = 320$.

2. Lists and exterior shortage

Arbitrary lists of size at least $r/61$ in any subset of the ordinary rows admit independent representatives. Draw one point per list uniformly and independently. An edge event has probability at most $61^2/r^2$ and shares a row with at most $2\gamma\rho r^2 + 1$ events including itself, by the outside-part cut in (3). The symmetric

Lovász local lemma applies since

$$(2719/1000)\,61^2(2\gamma\rho_0 + 2^{-40}) < 1, \qquad \exp(1) < 2719/1000, \quad r \geq 2^{20}.$$

Let G consist of the exterior points of degree at most ρr in every ordinary row and let $\Delta = hr - |G|$. We claim $\Delta > cr$ for $c = 121/2000$. The reduction in §3 of Appendix G applies with the following verifications. First, $h+1$ independent points of G leave fraction at least $1 - g\rho - f > 1/61$ in each row. Second, $h \geq K - 1$, $h^2/r < 1/(RK^2) < 2^{-40}$. Thus the auxiliary part bound in that section is

$$(2 + 1/(K-1))c^2 + 3 \cdot 2^{-40} < (43/500)^2.$$

The auxiliary cliques, their $r/10$ lists and the attachment calculations in §3 use these bounds and $r \geq 2^{20}$.

For the temporary restriction of each ordinary clique, remove vertices of degree greater than $r/40$ to $V \setminus R_i$. At most $40y_i r$ vertices are lost. The number of points in $R_i \setminus C_i$ is $d_i + b_i \leq (g\rho + 1/256)r$. Hence the survivors have degree at most $r/32$ outside the original clique, and the surviving size is at least $255r/256$, since

$$1/40 + g\rho_0 + 1/256 < 1/32, \qquad (g + 40\gamma)\rho_0 < 1/256.$$

With $\theta = 93/250$, both $255/256 - \theta - f - 4/32$ and $255/256 - 2\theta - f$ exceed $1/61$. These are the two ordinary list sizes used by the extension in §3. Its contradiction gives the claimed shortage. For the circular-arc extension in §4, the available fraction is $1 - g\rho - 3/4 - f > 1/61$. This verifies the ordinary list condition at every use.

3. An edge bound for strong assignments

Return to the original ordinary cliques. There are $D + \Delta + 1$ exterior vertices outside G. As in the base manuscript, assign a vertex with degree greater than $r/2$ in at least two ordinary rows to two such rows. Assign a vertex with just one such row to it. The remaining vertices outside G are called moderate. In a row of deficit d, write b for the count of strong assignments. For any z exterior points of row degrees j_y, the original colors give

$$\sum_y (j_y + d) \leq (r-1)d + z(z+1)/2. \tag{4}$$

Indeed each of the $r-1$ other colors needs at least $z - d$ edges on the points and the clique together. Applying (4) to $2d+1$ strong assignments would give $r/2 < d+1$, excluded by (3). Thus $b \leq 2d$. With $z = b$, (4) also gives $\sum j_y \leq rd$, because $b(b+1)/2 - bd \leq b/2 \leq d$.

For a uniquely assigned vertex, write $q = j_y/(r-d) > 1/2$. Draw uniformly in the clique. If the first point is a neighbor, retry once with probability 6ρ, independently and uniformly. A nonneighbor is obtained with probability $v = (1-q)(1+6\rho q)$. Use independent trials for the different senders. The degree plus deficit pays

$$j_y + d \;\geq\; r(1-v) + \rho r v + 2d(1-q).$$

The difference is $(1-q)\{r[6\rho q(1-\rho) - \rho] - d\} \geq 0$, since $3(1-\rho_0) - 1 > g$. Each assignment of a doubly assigned vertex pays at least $r/2 + 2d(1-q)$. Denote the sum of these last $2d(1-q)$ contributions in one row by J_d. It is nonnegative and $J_d \geq 2d(b - d/(1-d/r))$ by the bound on $\sum j_y$.

For **every** $v_0 \geq 0$, with $\lambda = 501/1000$, we have the row inequality

$$b^2/2 + (b-d)v_0 - J_d \leq \lambda d^2 + v_0^2/2. \tag{5}$$

For $b \leq d$ this follows immediately. Otherwise write $b = (1+t)d$, $0 < t \leq 1$. The left side is at most

$$(1-t)^2 d^2/2 + tdv_0 + \frac{2(d/r)}{1-d/r}\, d^2.$$

The first two terms are bounded by $(1-t)d^2/2 + t(d^2+v_0^2)/2$. The last coefficient is below $1/1000$ by (3). This proves (5), including $d=0$ in the first branch. Lean checks the retry inequality and (5) for real parameters under the stated bounds. Lean source

All moderate vertices and all successful unique senders are eligible. Let A be their expected count and $L = D + \Delta + 1 - A$. Apply (4) to the strong assignments, sum (5) with $v_0 = 0$, and use $\sum b/2 \le D$. This gives $rL \le rD + \lambda\sigma$. Mark eligible vertices independently with probability $p = \min(1, \Delta/A)$, which is defined since $A \ge \Delta + 1 - \lambda w > 0$. Write $W = G \cup Z$ for G and the marks, $m = \mathbb{E}|Z|$, $u = \Delta - m$. Then

$$0 \le u \le \lambda w, \qquad (c - \lambda\kappa\rho)r \le m \le sr. \tag{6}$$

4. Counting the repairs

Use the three representative conflicts and the circular arcs of §4 of Appendix G. The arcs still have length four times the load on $3r$ positions. Loads are at most $3r/4$, and §2 above verifies that the repaired graph on W has independence at most h. Write $E_i = \mathbb{E}e_J(C_i, W)$, F_i for the expected incidences into C_i from representatives in other rows with multiplicity, n_i for the expected count of loads greater than ρr in row i, and N_i for the number of uniquely assigned strong senders there. Put

$$Q = \sum n_i(E_i + F_i)/r, \quad Q_F = \sum n_i F_i/r, \quad T = \sum N_i E_i/r, \quad B_n = \sum n_i d_i/r.$$

At every thinning probability a particular point has representative probability at most j/r per sender in its row, with $j = 1001/1000$, because $(1+6\rho_0)/(1-g\rho_0) < j$. Since $N_k \le 2d_k$ and $2jg\rho_0 < 1/512$, we get $F_i \le y_i r^2/512$.

A marked sender from $R_i \setminus C_i$ counts at most once towards n_i. For the senders outside R_i, their own degrees into C_i sum to at most $y_i r^2$, and their representatives add at most F_i in expectation. Consequently, for $M = 8591/1000$,

$$\frac{n_i}{r} \le x_i + (b_i)_+/r + \frac{513y_i}{512\rho}, \qquad \sum n_i \le Msr.$$

Here $\sum(x_i + (b_i)_+/r) \le 5$ and $\sum y_i \le 2$. Also $\sum N_i \le 9r$, $\sum N_i d_i \le 2\sigma$, and $\rho r n_i \le E_i + F_i$, so $B_n \le \sqrt{\kappa Q}$. The two-clique bound $e_J(C_i, C_k) \le r + d_i d_k$ now gives $F_i \le 10r + 2.01wd_i$ and $Q_F \le 10Msr + 2.01w\sqrt{\kappa Q}$.

To count representative conflicts, remove from E_i the contribution E_i^0 of the marked unique senders of this same row. If their expected number is t_i, then $E_i^0 \ge rt_i/2$. Their mutual conflicts cost at most $t_i^2/2$ by independence. Representative hits on exterior points outside this group cost at most $jN_i(E_i - E_i^0)/r$, by independence across senders. Since $t_i \le N_i$, the sum of these two costs is at most jN_iE_i/r. Different-row representative pairs cost at most $243r + 2.01w^2$, by the two-clique bound and the two sums for N_i above. The arc-pair bound from the base manuscript is $4Q/3$. For the total count P_* of distinct added edges this proves

$$\mathbb{E}P_* \le (4/3)Q + jT + 2.01w^2 + 243r. \tag{7}$$

5. The expected cut and the contradiction

The signed cut identity in §5 of Appendix G gives, for rows tested independently of the repairs with probabilities π_i,

$$\sum \pi_i E_i \le \mathbb{E}P_* + r\mathbb{E}\binom{H}{2} + \tfrac{1}{2}(B_\pi + u)^2 + m/2,$$

where $H = h + \#\{\text{tested rows}\}$, $B_\pi = \sum \pi_i d_i$. Take $a = 54/25$, $b = 13/10$, $Y = 13/50$ and $\pi_i = (an_i + bN_i)/r$. The pointwise bound from §4 gives $\pi_i \le a(513\gamma/512 + 1/256 + g\rho_0) + 2bg\rho_0 < 1$. Set $V = aM + 9b/K$ and $B_0 = ((1+V)^2 + V/K)/2$. Then $\sum \pi_i \le Vs$, $\mathbb{E}\binom{H}{2} \le B_0 s^2$, and $B_\pi \le a\sqrt{\kappa Q} + 2bw$.

For reproducibility, put $t = 2b + \lambda$, $\chi = 2.01$, $k = 17/5$. The left side of the cut bound is $a(Q - Q_F) + bT$.

Substitute (6), (7) and the bound on Q_F, and bound the resulting mixed term by

$$a(t+\chi)w\sqrt{\kappa Q} \le YQ + \frac{a^2(t+\chi)^2\kappa}{4Y}w^2.$$

The coefficients remaining on the left are $a - 4/3 - a^2\kappa/2 - Y$ for Q and $b - j$ for T. Multiplication by k makes them at least $4/3$ and j respectively. Adding the two direct terms of (7) gives

$$(4/3)Q + jT + \chi w^2 + 243r \le 55w^2 + 651s^2 r. \tag{8}$$

The two coefficient bounds used here are

$$k(\chi + t^2/2 + a^2(t+\chi)^2\kappa/(4Y)) + \chi < 55,$$
$$k(B_0 + (10Ma + 1/2)/K + 243/K^2) + 243/K^2 < 651.$$

Let $\eta = 1/(Rf^2)$. By (3), $s^2 r \le \eta\rho^2 r^2$ and $55\kappa^2 + 651\eta < 33/100$. For a single tested row the last two terms of the signed cut estimate sharpen to $ud_i + u^2/2 + m/2$, since no pair of rows is tested. Here $H = h + 1 = s$. Hence

$$\frac{E_i}{\rho^2 r^2} \le 33/100 + \lambda\kappa g + (\lambda\kappa)^2/2 + \eta(1/2 + 1/(2K)) < 3/8. \tag{9}$$

If $A > \Delta$, oversample using $z' = \min(A - \Delta, \Delta/128, r/16)$ and $p' = (\Delta + z')/A < 2p$. The heavy counts and representative incidences scale linearly, G is fixed, and therefore $Q' \le 4Q$, $T' \le 2T$. By (7)–(8), the expected repair cost is at most $4(33/100)\rho^2 r^2$. The oversampling Turán floor from §5 of Appendix G gives at least $rz'/2 - rs^2$, using $h \le r/4$, $z' \le r/16$, and variance at most sr. It follows that $z'/r \le (8 \cdot 33/100 + 2\eta)\rho^2 < (8/3)\rho^2$. This is below both $c/128$ and $1/16$, so $z' = A - \Delta$. Including $A \le \Delta$, we have $L \ge D - (8/3)\rho^2 r$.

Assign each moderate sender in advance to a row of degree above ρr. The count V_i of its marked assignments there has mean v_i, variance at most v_i, and $\sum v_i \le m$. Since $\rho r v_i \le E_i$, (9) gives $v_i \le (3/8)\rho r$. Apply (4) to the b'_i strong assignments and the V_i marked moderate assignments in each row. On the left the strong senders supply rL, the marks supply ρrm, and the moderate deficits supply $\sum d_i v_i$, in addition to the strong surplus J_{d_i}. On the right $\sum b'_i/2 \le D$ absorbs the subtraction of D from $(r-1)D$. Moving the deficits and surplus across and applying (5) with $v_0 = v_i$ gives

$$rL + \rho rm \le rD + \lambda\sigma + \sum v_i^2 + m.$$

Indeed the variance and the linear moderate term cost at most v_i per row. The moderate square contributes $v_i^2/2$ directly and (5) contributes at most another $v_i^2/2$.

Using $\sum v_i^2 \le (3/8)\rho rm$, (6) and $m \le sr$, this would imply

$$(1 - 3/8)(c - \lambda\kappa\rho) \le \lambda\kappa + (8/3)\rho + 1/(fRs^2).$$

At $s = K$ the left side exceeds the right side by more than $1/10000$. For larger s the left side increases and the right side decreases. This contradiction proves the theorem.

Appendix G. Stability, shortage and the original cut

1. Extracting target-color cliques

Assume a counterexample and choose a sparsest color J. Every color graph has independence number at most r, and every monochromatic clique has order at most r. Edges, degrees and independence below refer to J unless a different color is specified. Put $B(x) = x(x-1)/2$. Write $p_k(n)$ for the minimum number of edges of a graph of order n with independence number at most k. By Turán's theorem this is the sum of B on k balanced integer part sizes.

Füredi's stability theorem applied to the complement of J gives r parts R_i on all r^2+1 points. The number μ of internal pairs missing J is at most $\eta = e_J(V) - rB(r) - r \le B(r)$, since the color has at most $r(r^2+1)/2$ edges. Put $b_i = |R_i| - r,\ O = \sum (b_i)_+,\ V_2 = \sum b_i^2$. Each other hue needs at least $(b_i)_+$ edges in R_i, by deleting one endpoint per edge. Since $\sum b_i = 1$ and $V_2 \ge 1$, the number X of target edges between parts satisfies

$$O \le \mu/(r-1) \le r/2, \qquad X = \eta + \mu - (V_2 - 1)/2 \le r(r-1).$$

Füredi's stability theorem

Delete points of current total interpart degree greater than $r/4$, one at a time. Each deletion removes fresh cross edges, so the number P deleted satisfies $rP/4 \le X$. Trim parts above r to r. The sum of the deficits from r over all slots is at most $P + O - 1 \le 9r/2$.

Take a maximal family of q pairs, each inside a different slot, whose $2q$ endpoints are jointly independent. In unchosen slots discard their neighbors, at most $qr/2$ points. Each unchosen slot's survivors form a clique by maximality. Let f_0 unchosen slots now have size below $r/2$. The other unchosen slots admit independent representatives by Haxell's theorem, since their interlist degree is at most $r/4$. With the endpoints this gives $r + q - f_0 \le r$. Omitting the short slots therefore pays for all the neighbor deletions in the deficit sum, since $f_0 r/2 \ge qr/2$. Haxell's independent transversal theorem

To bound the number of omitted slots, at most 36 were initially below $7r/8$ after trimming. Among initially larger unchosen slots, at most $4q/5$ fall below $r/4$ after discarding the neighbors. On lists still of size at least $r/4$, temporarily prune current interlist degrees above $r/16$. At most $16X/r \le 16r$ points are lost and at most 128 more lists fall below $r/8$. Haxell extends the original endpoints on the remaining lists. Thus $q \le 36 + 4q/5 + 128$, giving $q \le 820$. Also $f_0 \le 9 + q$, by the total deficit after neighbor deletion. Undo the temporary pruning and set aside the chosen and short slots. Their count is $e \le 1649$.

For each resulting clique C_i put $d_i = r - |C_i|,\ \ell_i = e_J(C_i, V \setminus C_i),\ x_i = d_i/r$, and $a_i = \max(x_i, \ell_i/r^2)$. We claim

$$\sum x_i \le 9/2, \qquad \sum a_i \le 11/2. \tag{1}$$

For $b_i > 0$, at least $(r-1)b_i$ internal nontarget pairs avoid the clique interior. Subtract these from $B(r+b_i) - B(r-d_i)$ to bound the cut inside R_i by $rd_i + B(b_i+1) - B(d_i+1)$. For $b_i \le 0$, it is at most rd_i. The original $2X$ cross incidences include at least $rP/4$ at removed points. Since $B((b_i)_+ + 1) \le b_i^2$, the deficit estimate gives

$$r^2 \sum a_i \le r(P + O - 1) + V_2 + 2X - rP/4 \le 5X + rO - r + V_2 \le (11/2)r^2.$$

For the last bound use $5X + V_2 \le 10\eta + 1$. Every retained point has at most $r/4$ target neighbors in the other retained cliques together.

2. Selecting the width by a weight threshold

Set

$$f = 6/25, \quad \gamma = 14/25, \quad \kappa = 113/5000, \quad \alpha = \gamma f, \quad \beta^2 = \kappa f, \quad k_0 = 2^{14}.$$

For a level $z > 0$, let $s(z) = e + k_0 + 1 + \#\{i : a_i > z\}$. Define

$$A(z) = s(z)z, \qquad C(z) = \sum_{a_i \le z} x_i^2/z.$$

Integrate $A/\alpha + \alpha C/(4\beta^2)$ against dz/z between $\alpha 2^{-115}$ and $\alpha 2^{-14}$. Each row contributes at most a_i to the integral of $z\#\{a_i > z\}$, and at most $x_i^2/a_i \le x_i$ to the integral of C. A zero weight contributes zero. By (1) the integral is at most

$$\frac{11}{2\alpha} + \frac{9\gamma}{8\kappa} + \frac{k_0 + 1650}{k_0} < 69.90 < 101 \log 2.$$

Hence at some level the integrand is below one. At this level $A < \alpha$ and $AC < \beta^2$ by the arithmetic–geometric mean inequality. In particular $s = s(z) < 2^{115}$. Omit the rows above the level and any k_0 additional rows. There are enough, since $s < r$. Put $h = s - 1,\ \rho = f/s$. For the $r - h$ remaining ordinary cliques, define $D = \sum d_i,\ \sigma = \sum d_i^2,\ w = \sigma/r$. Then

$$d_i \le \gamma\rho r, \quad \ell_i \le \gamma\rho r^2, \quad w \le \kappa\rho r, \quad D \le 9r/2, \quad \sum \ell_i \le 11r^2/2. \tag{2}$$

We have $s \ge 2^{14}$ and $r \ge Rs^4$ with $R = 2^{23}$, because $23 + 4 \cdot 115 = 483$.

Arbitrary lists of size at least $r/110$ in any subset of ordinary rows have independent representatives. Draw independently and uniformly. Every target-edge event has probability at most $110^2/r^2$. The events sharing either of its rows, including itself, number at most $2\gamma\rho r^2 + 1$. The symmetric Lovász local lemma applies since $4 \cdot 110^2(2\gamma\rho + 1/r^2) < 1$. All local-lemma applications below have these same ordinary lists.

The explicit affine square in the preceding draft had full rows with actual cut weights bounded by $1/(26i)$ in row i. These rows already pass the present pointwise test at k_0, because $1/26 < \alpha$, and their squared deficits vanish. A scalar obstruction remains. Take $x_i = a_i = 3/(40i)$ for $1 \le i \le 2^{80}$. Its mass is below $9/2$. After any $h \le 2^{74}$ deletions the least possible squared tail is obtained by removing the largest terms. Integration gives

$$(h+1)\sum_{i>h}^{2^{80}} x_i^2 \ge (3/40)^2(1 - (h+1)/(2^{80}+1)) > \beta^2.$$

This example establishes a limitation of (1) alone. Whether a coloring on $r^2 + 1$ points realizes a profile of this strength is open here.

3. A shortage of exterior vertices

Let G contain all exterior points having at most ρr target neighbors in each ordinary row. Write $\Delta = hr - |G|$. We prove

$$\Delta > cr, \qquad c = 121/2000. \tag{3}$$

An independent $h + 1$ points of G would leave at least $(1 - \gamma\rho - f)r > r/110$ choices per ordinary row and extend to an independent $(r + 1)$-set. Thus $\alpha_J(G) \le h$. If $|G| \ge hr + 1$, take $X_0 \subseteq G$ of order $hr + 1$. Otherwise suppose $0 \le \Delta \le cr$ and take $X_0 = G$. Write $|X_0| = hr + b$, where $b = 1$ or $-\Delta$.

For signed integers v, extend $p_k(v)$ by the same balanced-integer-parts formula. In particular $p_k(-v) = p_k(v) + v$ for $v \ge 0$, and $p_k(kt + v) = kB(t) + tv + p_k(v)$. Applying the target and nontarget Turán floors on X_0 gives

$$0 \le \eta_0 = e_J(X_0) - p_h(hr + b) \le rB(h) + c^2r^2/2.$$

Indeed the upper bound before simplification is $rB(h) - (r - h)b + B(b) - p_h(b) - (r - 1)p_r(b)$. With $\xi = c^2 + 2^{-40}$ we get $2\eta_0 \le \xi r^2$, since $h^2/r < 2^{-40}$.

Apply Füredi to the complement of the target on X_0 with h parts. Write $V_0 = \sum(|R| - r)^2$, V_{bal} for

the balanced value and X_a for the cross edges. Expansion of the internal binomials, using at most η_0 internal missing pairs, gives $2X_a + V_0 \le M^2 = 4\eta_0 + V_{\mathrm{bal}}$ and $X_a \le 2\eta_0$. If $b = hq + j,\ 0 \le j < h$, then $V_{\mathrm{bal}} = b^2/h + j(1 - j/h)$. Consequently $M^2/r^2 \le (2 + 2^{-14})c^2 + 3 \cdot 2^{-40} < \delta^2$, with $\delta = 43/500$. Prune points of current interpart degree above τr, where $\tau = 9/100 > \delta$. For each part with original positive deficit $d = (r - |R|)_+$, its final deficit is at most

$$d + X_a/(\tau r) \le d + (M^2 - d^2)/(2\tau r) \le M.$$

If two survivors in a part were nonadjacent, every other part would retain at least $r/2$ common nonneighbors. Random representatives from them have at most $4\xi < 1$ target edges in expectation. This would violate the target independence cap h. Thus there are h auxiliary cliques of sizes at least $(1 - \delta)r$. Arbitrary lists of size at least $r/10$ in them admit independent representatives by the same expectation bound, now $100\xi < 1$.

In the positive case use just these auxiliary rows on X_0. In the other case use them together with the ordinary rows on all vertices. For this step temporarily remove from each ordinary clique the points with degree above $r/32$ outside its original clique. At most $32\ell_i/r$ are removed, leaving at least $(1 - 33\gamma\rho)r \ge 255r/256$. In either case there are $pr + 1$ vertices and p rows, with target independence cap p. Put $\theta = 93/250$. Call an attachment large when its target degree to a row exceeds θr.

Prescribe either one exterior point and at most four mutually compatible representatives in different rows, or two independent exterior points. Suppose all exterior attachments to the rows still to be filled are small. Rows may be deliberately omitted. Fill auxiliary rows first. In the first case every prescribed representative forbids at most $\max(\tau, 1/32)r$ per auxiliary row. In the two cases the remaining fractions are at least $1 - \delta - \theta - 4\tau$ and $1 - \delta - 2\theta$, both above $1/10$. In any ordinary row every auxiliary choice together costs at most $h\rho r \le fr$, because the auxiliary rows lie in G. The two available fractions are at least $255/256 - \theta - f - 4/32$ and $255/256 - 2\theta - f$, each above $1/110$. The list results therefore give the whole extension.

It follows that every exterior point has a large row. A point with a unique large row is complete to it, since one nonneighbor could be prescribed there. Two such points assigned to the same row are adjacent, since otherwise they could extend together while omitting it. Adjoin all these points to their respective cliques, giving deficits $v_i \le \delta r$. If P_0 exterior points remain, then $\sum v_i = P_0 - 1$, and every one has at least two large attachments to the rows before enlargement.

Two capacities will be used repeatedly. For any target clique C_i of size $r - v_i$, and any z exterior points Y,

$$\sum_{y \in Y} (d_J(y, C_i) + v_i) \le (r - 1)v_i + z + B(z). \tag{4}$$

Each nontarget hue needs at least $z - v_i$ edges on the union. Likewise two disjoint target cliques have at most $r - v_i - v_j + v_i v_j \le r + v_i v_j$ target edges between them, since each nontarget hue needs at least $r - v_i - v_j$ edges there.

Write $t = 133/500$. A row has fewer than $\lfloor tr \rfloor$ remaining large arrivals. Otherwise (4), applied to that many points at the enlarged row, would give

$$\theta(t - 1/r) < (1 - t + 1/r)\delta + t^2/2 + t/(2r),$$

which fails for $r \ge 2^{20}$. Each row's term $z + B(z)$ is therefore at most $z(tr + 1)/2$.

Consider a point with exactly two large rows, and let u_1, u_2 be its two degrees divided by r before enlargement. If $u_1 + u_2 \le 1 + t + 1/r$, then both are at least θ and both are at most $1 + t + 1/r - \theta$. Its two nonneighbor lists have product at least

$$r^2(1 - \delta - \theta)(\theta - \delta - t - 1/r) > r + \delta^2 r^2.$$

Indeed the product of the two lower bounds $1-\delta-u_i$ is smallest at maximal sum, then at an endpoint by concavity. The factors there are positive. The two-clique cut bound gives a compatible pair, which would extend with this point by the preceding extension. Thus its two arrivals together, after paying $(tr+1)/2$ apiece, contribute more than r.

For exactly $j = 3, 4$ large rows, use $\lambda_3 = 153/1000$, $\lambda_4 = 43/200$. If all nonneighbor lists had size at least $\lambda_j r$, independent random representatives would meet at most $B(j)(1/r+\delta^2)/\lambda_j^2 < 1$ target edges in expectation and could extend with the point. One attachment exceeds $(1-\delta-\lambda_j)r$. The net charge in units of r exceeds $1-\delta-\lambda_j+(j-1)\theta-j(t+1/r)/2 > 1$. For $j \geq 5$, use $5(\theta-(t+1/r)/2) > 1$. Applying (4) to these arrivals at all enlarged rows now gives $rP_0 \leq r\sum v_i$, impossible since $\sum v_i = P_0 - 1$. This proves (3).

4. Sampling and circular arcs

Use the original ordinary cliques. There are $D+\Delta+1$ exterior points outside G, called senders. A strong attachment is one of degree greater than $r/2$. Assign a sender with at least two strong attachments to two of them, a sender with a unique strong attachment to that row, and call the others moderate. At row i, let b_i' be its strong assignment count and N_i its count of unique strong senders. Applied to $2d_i+1$ strong arrivals, (4) would require $r/2 < d_i + 1$. Thus $N_i \leq b_i' \leq 2d_i$.

At a unique strong sender let q be its target-neighbor fraction in its row. Draw a uniform representative there. If the first draw hits a neighbor, make a second independent uniform draw with probability $\lambda = 4\rho$. Keep the first nonneighbor obtained. The success probability is $v = (1-q)(1+\lambda q)$. All successes and all moderate senders are eligible. Trials are independent between senders. The strong degree plus the row deficit is at least rq. Since $q \geq 1/2$ and $\lambda q \geq \rho(1+\lambda q)$, it pays at least $r(1-v)+\rho r v$. The two assignments of a multiply strong sender pay at least r together.

Let A be the expected eligible count and $L = D+\Delta+1-A$. Summing (4) on the strong assignments gives $rL \leq rD + 2\sigma$, using $\sum b_i'/2 \leq D$. Hence $A \geq \Delta+1-2w > 0$. Mark eligible senders independently with probability $p = \min(1, \Delta/A)$. Put Z for the marks, $W = G \cup Z$, $m = \mathbb{E}|Z|$, $u = \Delta - m$. Then

$$0 \leq u \leq 2w, \qquad (c - 2\kappa\rho)r \leq m \leq hr. \tag{5}$$

Add edges on W to impose three conflicts. Two marks bringing representatives in the same row conflict. A representative that sees another exterior point creates a conflict between their exterior points. Representatives in different rows that see one another also create a conflict. For any other row i, the load t_{xi} of a marked sender is its degree into C_i plus its representative's degree there, omitting the latter for moderate senders. Give load zero in the sender's represented row and for unmarked senders. Every positive load is at most $3r/4$, by the definition of a unique strong row and the $r/4$ interpart degree bound.

In each row give every mark with $t_{xi} > \rho r$ an arc of $4t_{xi}$ consecutive positions on a circle of $3r$ positions, with independent uniform starts. Intersecting arcs create an edge between their senders. In an independent $h+1$-set of the repaired graph, the heavy loads in an unrepresented row sum to at most $3r/4$, by disjointness of their arcs. The other points and loads together cost at most $(h+1)\rho r = fr$. The available fraction $1-\gamma\rho-3/4-f$ exceeds $1/110$. Thus Section 2 fills the unrepresented rows. Together with the selected exterior points and their compatible representatives this would give $r+1$ independent vertices. The repaired graph has independence at most h for every sample and every choice of starts.

Let $E_i = \mathbb{E}e_J(C_i, W)$, let F_i be the expected sum of degrees into C_i from representatives in other rows with multiplicity, and let n_i be the expected number of heavy loads in row i. Write

$$Q = \sum n_i(E_i+F_i)/r, \quad Q_F = \sum n_i F_i/r, \quad T = \sum N_i E_i/r, \quad B_n = \sum n_i d_i/r.$$

Two arcs of lengths a, b intersect for at most $a+b$ relative starts, since the start difference must be a

difference of two positions in the length intervals. Conditional on the sender trials, the collision probability of two heavy arcs is at most $4(t_{xi}+t_{yi})/(3r)$. Sum over unordered pairs and expand into ordered terms. For distinct senders the heavy indicator of one is independent of the load of the other. Also $\sum_x \mathbb{E}t_{xi} \le E_i+F_i$. Thus the expected cost of the arc edges is at most $4Q/3$. Counting only pairs of distinct senders avoids a diagonal contribution.

A specific representative has marginal probability at most $(1+4\rho)/|C_j| \le 1.005/r$ for a particular sender in row j, at any thinning probability. Hence $F_i \le \ell_i/512$ by $N_j \le 2d_j$. Since $\rho r n_i \le E_i + F_i$ and $E_i \le \ell_i$, (2) gives $\sum n_i \le 23sr$, $\sum N_i \le 9r$, $\sum n_i^2 \le Q/\rho$, and $B_n \le \sqrt{\kappa Q}$. The two-clique bound also gives

$$F_i \le (1.005/r)\sum_{j\ne i} N_j(r+d_jd_i) \le 10r + 2.02wd_i, \qquad Q_F \le 230sr + 2.02w\sqrt{\kappa Q}. \tag{6}$$

Let P_* count distinct added edges. The conflicts from a representative hitting another exterior point cost at most $1.005T$, by independence from each other sender and deterministic membership of G. A representative misses its own sender. If b unique strong senders are marked in expectation in a row, their same-row conflicts cost at most $b^2/2$ by independence. We have $b \le (1+4\rho)N_i/2$ by the success probability, and $br/2 \le E_i$. The sum of these first two costs is at most $1.51T$. Distinct-row representative conflicts cost at most

$$\frac{1.005^2}{r^2}\sum_{i<j} N_iN_j(r+d_id_j) \le 243r + 2.1w^2,$$

by $\sum N_i \le 9r$ and $\sum N_id_i \le 2\sigma$. Therefore at every thinning probability,

$$\mathbb{E}P_* \le (4/3)Q + 1.51T + 2.1w^2 + 243r. \tag{7}$$

5. Charging repairs to the actual colors

Take a particular sample and a set I of tested rows. Set $H = h+|I| \le r$, $v = hr-|W|$, $D_I = \sum_{i\in I} d_i$, $z = v + D_I$. On $U = W \cup \bigcup_{i\in I} C_i$, let $X_I = e_J(W, \bigcup_{i\in I} C_i)$, and let Y count target edges between tested cliques. Let a be the excess of the repaired edge count on W over $p_h(|W|)$, and b the sum of the excesses of all nontarget hues on U over $p_r(|U|)$. They are nonnegative by the independence caps. Partitioning all pairs on U and expanding the signed balanced-part formula of Section 3 gives

$$P_* + rB(H) + \sum_{i<j\in I} d_id_j + vD_I + B(v) - X_I = Y + a + b + (H-1)z + p_h(v) + (r-1)p_r(z).$$

For example, before expansion $X_I + Y = B(|U|) - \sum_{i\in I} B(r-d_i) - p_h(|W|) + P_* - a - (r-1)p_r(|U|) - b$. The right side of the displayed identity is nonnegative even for negative z, since $p_r(z) \ge -z$ then. All Turán applications concern actual graphs of nonnegative order.

Test rows independently with probabilities π_i, independently of the sender trials and arc starts. With $B_\pi = \sum \pi_i d_i$, the expected cut satisfies

$$\sum \pi_i E_i \le \mathbb{E}P_* + r\mathbb{E}B(H) + (B_\pi + u)^2/2 + m/2. \tag{8}$$

Indeed $\mathbb{E}v = u$, $\mathrm{Var}(v) \le m$ by independent marking outcomes. For a single fixed row the last two terms can instead be bounded by $ud_i + u^2/2 + m/2$, because the pair-of-rows sum vanishes.

Take $a_0 = 89/50$ and $\pi_i = (a_0n_i + 2N_i)/r$. These are valid since $\pi_i \le a_0(513/512)\gamma + 4\gamma\rho < 1$. Also $\sum \pi_i \le 42s$, so $\mathbb{E}B(H) \le ((43s)^2 + 42s)/2 < 925s^2$. The left side of (8) is $a_0(Q-Q_F) + 2T$, while $B_\pi \le a_0\sqrt{\kappa Q} + 4w$. Combining (5)–(8), expand the square and absorb its mixed term together with the

mixed term from $a_0 Q_F$, using

$$a_0(6+2.02)w\sqrt{\kappa Q} \le Q/10 + (5/2)[a_0(6+2.02)]^2\kappa w^2.$$

This yields

$$(a_0 - 4/3 - a_0^2\kappa/2 - 1/10)Q + (2-1.51)T \le (2.1+18+(5/2)[8.02a_0]^2\kappa)w^2 + 928s^2 r.$$

Multiplying by 4.29 dominates $(4/3)Q + 1.51T$. Using (7) and then (8) for one row gives

$$\mathbb{E}P_* \le 138w^2 + 3992s^2 r, \qquad E_i \le 140w^2 + 2wd_i + 3993s^2 r. \tag{9}$$

If $A > \Delta$, let $z' = \min(A-\Delta, \Delta/128, r/16)$ and use $p' = (\Delta + z')/A < 2p$. Then $Q' \le 4Q$, $T' \le 2T$, by linear scaling of each sender trial and the fixed set G. Applying (7) and the same bound on $(4/3)Q + 1.51T$ limits $\mathbb{E}P'_*$ to $4(138w^2 + 3992s^2 r)$. Conversely, if the oversample has size $hr + b'$, the target floor on the repaired graph and the actual nontarget floors give

$$P'_* \ge (r-h)b' - B(b') - rB(h).$$

For $b' \ge 0$, expanding the floors adds the nonnegative terms $p_h(b') + (r-1)p_r(b')$ to the right side. For $b' < 0$ the displayed right side is nonpositive. Since $\mathbb{E}b' = z' \le r/16$, $h \le r/4$, and $\mathrm{Var}(b') \le \Delta + z' \le sr$, this gives $\mathbb{E}P'_* \ge rz'/2 - rs^2$. Hence

$$z' \le 1110w^2/r + 32000s^2.$$

Divided by r, this is at most $1110\kappa^2\rho^2 + 32000/(Rs^2) < \min(c/128, 1/16)$. By (3), only $A - \Delta$ can attain the minimum. Also allowing $A \le \Delta$, we obtain

$$L \ge D - 1110w^2/r - 32000s^2. \tag{10}$$

6. The capacity inequality

Assign each moderate sender in advance to a row where its degree exceeds ρr. Let v_i count those marked in the base sample at row i, with mean $\bar{v}_i$. Since $\rho r\bar{v}_i \le E_i$, (2) and (9) give

$$\bar{v}_i/(\rho r) \le 140\kappa^2 + 2\kappa\gamma + 3993/(f^2 R) < \epsilon, \qquad \epsilon = 211/2000.$$

We have $\mathrm{Var}(v_i) \le \bar{v}_i$ and $\sum \bar{v}_i \le m$. In (4) use all strong assignments and all marked moderate assignments. The expected charge on the left is at least $rL + \rho rm + \sum d_i\bar{v}_i$. This keeps the deficit paid by each moderate assignment. The unique strong surplus from Section 4 pays ρr for an expected successful mark, since $p \le 1$.

On expanding $(b'_i + v_i)(b'_i + v_i + 1)/2$, the strong squares cost at most 2σ, and the strong linear terms cancel the subtraction of D in $(r-1)D$. The mixed terms minus the moderate deficit charge cost at most $\sum d_i\bar{v}_i$, because $b'_i \le 2d_i$. This is bounded by $\sqrt{\sigma\epsilon\rho rm}$ using Cauchy–Schwarz. The moderate squares and linear terms cost at most $\epsilon\rho rm/2 + m$. By (10),

$$(1-\epsilon/2)\rho rm \le 2wr + \sqrt{\sigma\epsilon\rho rm} + 1110w^2 + 32000s^2 r + m.$$

By (5), the square-root term divided by ρrm is at most $\sqrt{\epsilon\kappa/(c-2\kappa\rho)} < 199/1000$. Dividing by ρr^2 and using $r \ge Rs^4$, $m \le sr$, would force

$$(1 - \epsilon/2 - 199/1000)(c - 2\kappa\rho) \le 2\kappa + 1110\kappa^2\rho + 32001/(fRs).$$

At $s = 2^{14}$ the left side exceeds the right by more than $1/20000$. The margin increases with s. This

contradiction proves Theorem 1.

Appendix H. Extracting cliques on all vertices

1. Selecting and restoring cliques on the full vertex set

Suppose every $(r+1)$-set sees all r colors. Each color graph has independence number at most r, and any monochromatic clique has size at most r. Write J for a sparsest color and $B(v) = v(v-1)/2$. Unless specified, edges, degrees, cliques and independent sets refer to J.

Apply Füredi's stability theorem to the complement of J on all r^2+1 vertices. It gives r parts R_i, with μ_i internal pairs missing J, such that

$$\mu = \sum \mu_i \le \eta = e_J(V) - rB(r) - r \le B(r).$$

The last bound uses $e_J(V) \le r(r^2+1)/2$. Put $b_i = |R_i| - r$, $O = \sum (b_i)_+$, $V_2 = \sum b_i^2$, and let X count interpart edges. Each of the other colors needs at least $(b_i)_+$ internal edges, since deleting one endpoint per edge leaves an independent set. Consequently

$$\sum b_i = 1, \quad O \le \mu/(r-1) \le r/2, \quad X = \eta + \mu - (V_2 - 1)/2 \le r(r-1).$$

Here $V_2 \ge 1$. Füredi's stability theorem

Stars into a single part. Set $U_i = \mu_i - (r-1)b_i$, so $U_i \ge 0$ and $\sum U_i = \mu - (r-1)$. If Z consists of m vertices outside R_i, each nontarget hue has at least $m + b_i$ edges on $R_i \cup Z$. Counting the available nontarget pairs gives

$$e_J(Z, R_i) \le U_i + (1 + b_i)m + B(m). \tag{1}$$

Qualify a part as a destination if $b_i \le r/1024$ and $U_i \le r^2/2048$. At most $512 + 1024 = 1536$ parts fail. Put $\tau = 33/200$ and $a = 81/500$. A qualified part has at most $U_i/(ar)$ exterior vertices of degree greater than τr into it. Indeed, for any $m \le \lfloor r/256 \rfloor$ such vertices, (1) gives

$$U_i > mr\,(\tau - 1/1024 - (m+1)/(2r)) \ge mar.$$

For $r \ge 2^{20}$, even $m = \lfloor r/256 \rfloor$ would exceed the qualifying budget. Applying the inequality to the actual number proves the claim (the empty set satisfies it directly).

First remove vertices one at a time when they send more than τr current edges to a single qualified foreign part. Then remove vertices whose current total interpart degree exceeds $r/4$, again one at a time. Let P_s, P_t count the respective removals. Counting original stars in the first stage and freshly removed edges across both stages gives

$$P_s \le \mu/(ar), \qquad \tau r P_s + r P_t/4 \le X. \tag{2}$$

Trim oversized parts to r, costing at most O vertices. The total deficit from r across these r slots is at most $P_s + P_t + O - 1 < 6r$, using (2) and $4 + (1 - 4\tau)/(2a) + 1/2 < 6$.

Extracting cliques. Across *all* slots take a maximal family of q nontarget pairs in distinct slots whose $2q$ endpoints are jointly independent. In unchosen slots delete neighbors of these endpoints, costing at most $qr/2$ vertices. The survivors of each unchosen slot form a clique by maximality. Let f of those slots now have size below $r/2$. The other unchosen slots, with size at least $r/2$ and interslot degree at most $r/4$, admit independent representatives by Haxell's theorem. Together with the endpoints they give $r + q - f \le r$. Hence $q \le f$. The f omitted short slots account for at least $fr/2$ final deficit. This pays for all $qr/2$ newly lost vertices. Thus the retained cliques have total deficit at most $P_s + P_t + O - 1$. In applications of Haxell's theorem first erase edges within each list. Independent transversal theorem, Theorem 1

For a bound on the *number* of omitted slots, at most 48 slots were below $7r/8$ before deleting the neighbors. At most $4q/5$ of the initially larger unchosen slots fall below $r/4$, since each loses more than $5r/8$. For

this count only, take the unchosen lists still of size at least $r/4$. Prune vertices of current interlist degree greater than $r/16$ in succession. This costs at most $16X/r \le 16r$ vertices, so at most 128 additional lists fall below $r/8$. Haxell extends the original pairs by one point from every remaining list. Hence

$$q \le 48 + 4q/5 + 128, \qquad q \le 880, \qquad f \le 12 + q \le 892.$$

The bound on f uses total deficit at most $6r + qr/2$ after deleting neighbors. Revert this temporary pruning. Set aside the q chosen slots, the f short slots and all unqualified destinations. The number e set aside is at most $3308 < 4096$.

For a remaining clique $C_i \subseteq R_i$, put $d_i = r - |C_i|$, $\ell_i = e_J(C_i, V \setminus C_i)$, and $a_i = \max(d_i/r, \ell_i/r^2)$. If $b_i > 0$, the compulsory $(r-1)b_i$ internal nontarget edges avoid $C_i \times C_i$. Subtracting them from $B(r+b_i) - B(r-d_i)$ bounds the cut inside R_i by $rd_i + B(b_i+1) - B(d_i+1)$. For $b_i \le 0$ the bound rd_i suffices. Summing the bounds on $\max(rd_i, \ell_i)$ uses at most V_2 for the terms $B((b_i)_+ + 1)$. At least $\tau r P_s + r P_t/4$ of the original $2X$ cross incidences had a removed endpoint. Thus

$$\begin{aligned} r^2 \sum a_i &\le r(P_s + P_t + O - 1) + V_2 + 2X - \tau r P_s - r P_t/4 \\ &\le 5X + (1 - 4\tau) r P_s + rO - r + V_2 \le (1061/162) r^2. \end{aligned} \tag{3}$$

Indeed $5X + V_2 \le 10\eta + 1$, $rP_s \le r^2/(2a)$, and $11/2 + (1-4\tau)/(2a) = 1061/162$.

Restoration. Within each retained part add back any vertex surviving the first pruning stage that is complete to its current clique and has full degree at most twice the clique's current size. Repeat until no such vertex remains. Each addition lowers the deficit. It changes the cut by $d_J(v) - 2|C_i| \le 0$ and preserves the other row cuts. Consequently (3) persists. Every point of a restored clique still has at most τr edges to *each* other retained clique because both consist of survivors of the first pruning stage and the destination qualified. The restriction on degree into a single row is what the grouped extension below uses.

Appendix I. The sharper cutoff and its Lean proof

Corollary. For every integer $r \geq R_* = 255 \cdot 2^{136}$, every r-coloring of the edges of K_{r^2+1} has $r+1$ vertices whose induced edges miss a color.

For a band with exponent b, replace S by $S' = S(255/256)^{1/4}$. Each band has $K' \leq 3942$ and $b \leq 17$. Hence $R_* > 2^{17}(8 \cdot 3942)^4$, giving $S' > 8K'$ as required in Section 2. Replacing the lower endpoint by $\ell' = \alpha/S'$ therefore preserves the endpoint argument. The five comparisons lose the same amount,

$$\log(S/K) - \log(S'/K) = \tfrac{1}{4}\log(1 + 1/255) < 1/1020.$$

Each previous certified margin was greater than $1/1000$. In fact the original rational certificates give a remaining margin greater than 0.00015316 in every band. The same successful-width argument now produces $K \leq s < S'$, so $2^b s^4 < 2^b (S')^4 = R_* \leq r$. The signed cut uses that inequality together with the same normalized bounds and the same 1,280 endpoints. This proves the corollary under the preceding combinatorial argument in the cutoff chapter.

Lean verifies a uniform reduced margin above $1/10000$ for every feasible pair of real pruning parameters and the identity for the fourth power of S'. Since $R_* = (255/256)2^{144}$, the cutoff decreases by about 0.39 percent. The first power of two covered is 2^{144}. Lean width proof · Rational replay

The formal statement

The theorem **erdos_617_of_ge_cutoff** expresses the conclusion using a coloring of unordered vertex pairs. The comparison only uses distinct vertices. This matches the public Formal Conjectures statement after restricting the number of colors to the proved range. **balancedColoringAt_of_ge_reduced_cutoff** uses the loop-free edge type in this project. **paper_cutoff** gives the manuscript's simpler power-of-two bound, and **finite_counterexample_orders** proves that the set of possible exceptional orders is finite. Lean statements · Original conjecture, p. 80 · Formal Conjectures statement

With the improved lower bound on the number of colors, the Lean theorem directly discharges the pinned Formal Conjectures statement with its original vertex type, edge coloring and conclusion.

The signed-cut inequality

1. Assume every set of $r+1$ vertices contains all colors. The width and exterior arguments select cliques and find a shortage of more than $3r/20$ exterior vertices with low attachment. Both sampling passes hold for those same cliques. Exterior shortage · Oversampling
2. At each destination, combine the strongly attached vertices with the marked, moderately attached vertices before bounding the edges into the clique. Write v_i for the expected moderate count at this row. The fluctuation of the actual count has mean zero and variance at most v_i. Its quadratic and linear costs together leave at most $v_i^2 + v_i$. The rows partition the moderate senders. The retry charge pays for each mark outside that partition, including the case of a unique strong destination. Summing proves the combined capacity inequality. Row bound and sum
3. The one-row cut controls $\sum_i v_i^2$. The second sampling pass controls the eligible mass above the shortage. Substitution forces the manuscript's signed expression (5) to be nonpositive at a specific normalized shortage l, provided the one-row coefficient is at most one. Inequality from the coloring
4. Within each width cell, the one-row coefficient is convex in l and the signed expression is concave. Exactly computed endpoint comparisons make the coefficient less than one and the signed expression strictly positive throughout the interval. This rules out the l supplied by the coloring. Interpolation proof · Exact rational computation · Transfer to the real parameters

The Lean certificate uses five bands and nine closed width intervals per band. In units of $1/128$, their boundaries are $0, 64, 72, 80, 96, 112, 120, 124, 126, 128$. The proof places every selected real width in at least one interval and bounds $w(1-w)$ even when its maximum occurs at $1/2$. The 90 endpoint tests are evaluated by Lean's kernel using exact rationals. The paper's separate Python checker uses 1,280 endpoints on a finer subdivision. Width cover and real bounds · Paper's checker

Verification commands

From the package directory containing **lean-toolchain**, install the pinned Lean toolchain through elan, then run

```
lake exe cache get
lake env python3 -B proof/replay_pruning_kernels.py cutoff --lean-only
```

This builds the cutoff sources in dependency order in a fresh directory and prints the axioms used by the final theorems. The project pins Lean 4.35.0 and mathlib commit **56a005855ea17e0b0958f73ec801ae140b0a1679**.

For the additional kernel replays, put **leanexport**, **leanchecker**, **lean4lean** and **nanoda_bin** on the Lake environment's path. The replay script checks the same export in all three kernels. Nanoda permits only propositional extensionality, classical choice and quotient soundness. The Python endpoint comparison can be run separately.

```
lake env python3 -B proof/replay_pruning_kernels.py cutoff
python3 -B proof/verify_retained_width.py
```

The formal theorem covers the large-color cutoff. The fixed-order arguments in the same package have their own verification records. Independent mathematical review of the manuscript remains pending.

Build instructions and verification record